Several rigidity features of von Neumann algebras

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Plusieurs aspects de rigidité des algèbres de von Neumann

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Plusieurs aspects de rigidité des algèbres de von Neumann

Résumé

Dans cette thèse je m'intéresse à des propriétés de rigidité de certaines constructions d'algèbres de von Neumann. Ces constructions relient la théorie des groupes et la théorie ergodique au monde des algèbres d'opérateurs. Il est donc naturel de s'interroger sur la force de ce lien et sur la possibilité d'un enrichissement mutuel dans ces différents domaines.

Le Chapitre II traite des actions Gaussiennes. Ce sont des actions de groupes discrets préservant une mesure de probabilité qui généralisent les actions de Bernoulli. Dans un premier temps, j'étudie les propriétés d'ergodicité de ces actions à partir d'une analyse de leurs algèbres de von Neumann (voir Theorem II.1.22 et Corollary II.2.16). Ensuite, je classifie les algèbres de von Neumann associées à certaines actions Gaussiennes, à isomorphisme près, en montrant un résultat de *W^*-superrigidité* (Theorem II.4.5). Ces résultats généralisent des travaux analogues sur les actions de Bernoulli ([KT08, CI10, Io11, IPV13]).

Dans le Chapitre III, j'étudie les produits libres amalgamés d'algèbres de von Neumann. Ce chapitre résulte d'une collaboration avec C. Houdayer et S. Raum. Nous analysons les *sous-algèbres de Cartan* de tels produits libres amalgamés. Nous déduisons notamment de notre analyse que le produit libre de deux algèbres de von Neumann n'est jamais obtenu à partir d'une action d'un groupe sur un espace mesuré.

Enfin, le Chapitre IV porte sur les algèbres de von Neumann associées à des groupes hyperboliques. Ce chapitre est obtenu en collaboration avec A. Carderi. Nous utilisons la géométrie des groupes hyperboliques pour fournir de nouveaux exemples de sous-algèbres maximales moyennables (mais de type I) dans des facteurs II_1.

Mots-clefs

Algèbres de von Neumann, actions Gaussiennes, ergodicité forte, W^*-superrigidité, sous-algèbres de Cartan, maximale moyennabilité.

Several rigidity features of von Neumann algebras

Abstract

The purpose of this dissertation is to put on light rigidity properties of several constructions of von Neumann algebras. These constructions relate group theory and ergodic theory to operator algebras.

In Chapter II, we study von Neumann algebras associated with measure-preserving actions of discrete groups : *Gaussian actions*. These actions are somehow a generalization of Bernoulli actions. We have two goals in this chapter. The first goal is to use the von Neumann algebra associated with an action as a tool to deduce properties of the initial action (see Corollary II.2.16). The second aim is to prove structural results and classification results for von Neumann algebras associated with Gaussian actions. The most striking rigidity result of the chapter is Theorem II.4.5, which states that in some cases the von Neumann algebra associated with a Gaussian action entirely remembers the action, up to conjugacy. Our results generalize similar results for Bernoulli actions ([KT08, CI10, Io11, IPV13]).

In Chapter III, we study amalgamated free products of von Neumann algebras. The content of this chapter is obtained in collaboration with C. Houdayer and S. Raum. We investigate *Cartan subalgebras* in such amalgamated free products. In particular, we deduce that the free product of two von Neumann algebras is never obtained as a group-measure space construction of a non-singular action of a discrete countable group on a measured space.

Finally, Chapter IV is concerned with von Neumann algebras associated with hyperbolic groups. The content of this chapter is obtained in collaboration with A. Carderi. We use the geometry of hyperbolic groups to provide new examples of maximal amenable (and yet type I) subalgebras in type II_1-factors.

Key-words

Von Neumann algebras, Gaussian actions, strong ergodicity, W*-superrigidity, Cartan subalgebras, maximal amenability.

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Chapitre I

Introduction générale

Dans cette thèse, j'étudie différentes constructions d'algèbres de von Neumann, en lien avec la théorie des groupes et la théorie ergodique. Je montre des résultats de rigidité de ces constructions, apportant un éclairage sur la classification des algèbres obtenues.

Cette introduction a pour but de présenter le contexte et les motivations de mon travail, avant d'expliquer les principaux résultats démontrés dans les chapitres II,III et IV. J'ai choisi de mettre en avant les concepts plutôt que la rigueur et le détail, dans l'idée que le lecteur souhaitant une présentation plus précise du domaine trouverait son bonheur dans des ouvrages déjà existant (par exemple [Fa09, Jon09, Po07b, Va10a, Io(12)b] ou plus classiquement [Co94, Di69, Ta02, Ta03]).

I.1 Algèbres de von Neumann

Soit H un espace de Hilbert complexe. L'ensemble $B(H)$ des opérateurs continus (ou bornés) $T : H \to H$ est une $*$-algèbre sur $\mathbb{C}$: on peut ajouter des opérateurs, les composer, et prendre leur adjoint. L'identité en est une unité, que l'on notera 1.

D'autre part, $B(H)$ est un espace topologique. Il admet la topologie donnée par la norme d'opérateurs, mais également la *topologie faible* : c'est la topologie la moins fine rendant continues les applications $T \mapsto \langle T\xi, \eta\rangle$, $\xi, \eta \in H$.

Une *algèbre de von Neumann* est une sous-$*$-algèbre de $B(H)$ contenant l'identité, et qui est faiblement fermée. La notion d'algèbre de von Neumann a vu le jour dans les années 1930, avec les travaux de Murray et von Neumann [MvN36], [MvN37], [vN40] et [MvN43] dans le but d'offrir à la physique quantique un cadre mathématique formel et unifié. Une autre de leurs motivations était de développer plus largement la théorie des représentations de groupes.

Le théorème fondateur de la théorie des algèbres de von Neumann est le *Théorème du bicommutant* de von Neumann qui caractérise les algèbres de von Neumann comme les sous-$*$-algèbres de $B(H)$ égales à leur bicommutant : $M = M''$. Rappelons que le commutant d'un ensemble $\mathcal{S} \subset B(H)$ est $\mathcal{S}' := \{T \in B(H)\,,\, ST = TS, \forall S \in \mathcal{S}\}$.

Enfin, de même que $B(H)$ peut être identifié au dual de l'espace $\mathcal{S}_1(H)$ des opérateurs à trace sur H, une algèbre de von Neumann est toujours isomorphe au dual d'un (unique) espace de Banach. Cette propriété caractérise les algèbres de von Neumann parmi les C^*-algèbres.

Ces caractérisations, de nature tantôt topologique, tantôt algébrique, tantôt analytique donnent accès à une multitude d'outils mathématiques. Cette richesse est une autre motivation pour étudier ces algèbres de von Neumann.

Présentons maintenant les principaux exemples étudiés dans cette thèse.

I.1.1 Exemples importants

Exemple I.1.1 (Algèbres abéliennes)**.** Si (X, μ) est un espace mesuré, l'algèbre $L^\infty(X, \mu)$ est une algèbre de von Neumann agissant sur l'espace de Hilbert $L^2(X, \mu)$ par multiplication. Cette algèbre est abélienne, et réciproquement toute algèbre de von Neumann abélienne est isomorphe à une telle algèbre de fonctions : c'est le théorème spectral.

Exemple I.1.2 (Algèbres de groupes)**.** Si Γ est un groupe discret dénombrable, on note $\lambda : \Gamma \to B(\ell^2(\Gamma))$ la représentation régulière gauche : $\lambda_g(\delta_h) = \delta_{gh}$, pour tous $g, h \in \Gamma$. L'algèbre de von Neumann $L\Gamma$ est par définition l'algèbre de von Neumann *engendrée* par $\{\lambda_g\, ,\, g \in \Gamma\}$, *i.e.* $L\Gamma = \{\lambda_g\, ,\, g \in \Gamma\}''$. C'est la plus petite algèbre de von Neumann sur $\ell^2(\Gamma)$ contenant les λ_g, $g \in \Gamma$.

Exemple I.1.3 (Group measure space construction)**.** Soit Γ un groupe discret dénombrable agissant de manière non-singulière sur (X, μ). Notons σ la représentation de Koopman sur $L^2(X, \mu)$ qui en découle et considérons l'espace de Hilbert $H = L^2(X, \mu) \otimes \ell^2(\Gamma)$. L'algèbre de von Neumann sur H engendrée par les opérateurs $f \otimes 1$, $f \in L^\infty(X, \mu)$ et $\sigma_g \otimes \lambda_g$, $g \in \Gamma$ est appelée la *group measure space construction* associée à l'action et notée $L^\infty(X, \mu) \rtimes \Gamma$.

Plus généralement, pour une action $\Gamma \curvearrowright N$ sur une algèbre de von Neumann N, on peut construire de manière analogue le *produit croisé* $N \rtimes \Gamma$ associé.

Étant données deux algèbres de von Neumann $M_1 \subset B(H_1)$ et $M_2 \subset B(H_2)$, il existe plusieurs procédés pour en construire une troisième.

- On peut former leur *produit tensoriel* $M_1 \overline{\otimes} M_2$ comme l'algèbre de von Neumann sur $H_1 \otimes H_2$ engendrée par les opérateurs $T_1 \otimes T_2$, $T_1 \in M_1$, $T_2 \in M_2$.
- On peut également considérer le produit libre amalgamé $M_1 *_B M_2$ au dessus d'une sous-algèbre de von Neumann commune $B \subset M_1, M_2$. Cette construction n'est possible que si B vérifie une certaine condition de "comparabilité" avec M_1 et M_2 (plus précisément il faut qu'il existe une *espérance conditionnelle* $M_i \to B$, pour $i = 1, 2$). Pour plus de détails sur cette construction, voir le Chapitre III.

Dans le cadre de l'exemple I.1.2, ces constructions sont les analogues du produit direct et du produit amalgamé de groupes.

Exemple I.1.4. Considérons Γ_1 et Γ_2 deux groupes discrets avec un sous-groupe commun Λ. Supposons que $\Gamma = \Gamma_1 *_\Lambda \Gamma_2$ agisse de manière non singulière sur un espace mesuré (X, μ). Alors on a un isomorphisme

$$L^\infty(X, \mu) \rtimes \Gamma \simeq (L^\infty(X, \mu) \rtimes \Gamma_1) *_{L^\infty(X,\mu) \rtimes \Lambda} (L^\infty(X, \mu) \rtimes \Gamma_2).$$

Avec tous ces exemples, la question de classification des algèbres de von Neumann s'impose : dans quels cas est-ce que ces constructions fournissent des algèbres isomorphes ?

I.1.2 Réduction aux facteurs II_1

Un *facteur* est une algèbre de von Neumann M dont le centre est restreint aux multiples de l'identité : $\mathcal{Z}(M) := M \cap M' = \mathbb{C}1$. Toute algèbre de von Neumann peut s'écrire comme intégrale directe (somme directe généralisée) de facteurs. Cette première observation permet de restreindre l'étude des algèbres de von Neumann générales à celle des facteurs.

Mais comment distinguer deux facteurs ?

Le théorème spectral montre qu'une algèbre de von Neumann contient beaucoup de projecteurs orthogonaux (ils engendrent un sous-espace vectoriel dense pour la norme d'opérateurs). Une étude comparative de ces projecteurs permet de distinguer trois grandes familles de facteurs : les facteurs de type I,II ou III.

Les facteurs de type I sont complètement classifiés : ils sont isomorphes à $B(H)$ pour un certain espace de Hilbert H (si $\dim H = n < \infty$, $B(H) \simeq M_n(\mathbb{C})$).

Les facteurs de type II se découpent en deux sous-types : les facteurs de type II_1 et II_∞. Les facteurs II_∞ s'écrivent comme produit tensoriel $M \overline{\otimes} B(H)$ avec M de type II_1 et $\dim H = \infty$. Les facteurs II_1 sont ceux qui possèdent une unique *trace fidèle*, c'est à dire une forme linéaire $\tau : M \to \mathbb{C}$ telle que $\tau(xx^*) = \tau(x^*x) \geq 0$ pour tout $x \in M$ et vérifiant $\tau(x^*x) = 0$ si et seulement si $x = 0$. En un sens, cette trace est l'analogue d'une mesure finie sur un espace mesurable.

Les facteurs de type III ont pendant longtemps été considérés comme hors de portée. C'est la théorie de Tomita-Takesaki et les travaux de Connes et Takesaki [Co73, Ta73, CT77] qui ont permis de mieux comprendre ces facteurs. Notamment, tout facteur de type III peut s'écrire comme le produit-croisé d'un facteur de type II par une action de $\mathbb{R}$.

Ainsi, l'étude des algèbres de von Neumann se ramène, au moins théoriquement, à celle des facteurs II_1. En pratique, il n'est pas immédiat de déduire des résultats sur des algèbres de von Neumann quelconques à partir de résultats analogues pour des facteurs II_1. C'est précisément ce que nous faisons dans le Chapitre III dans le cadre de produits libres amalgamés d'algèbres de von Neumann.

Illustrons cette classification en types I, II et III avec les Exemples I.1.2 et I.1.3.

- Si Γ est un groupe discret dénombrable, $L\Gamma$ est un facteur si et seulement si les classes de conjugaison non-triviales de Γ sont infinies (en abrégé : Γ est ICC). Dans ce cas c'est un facteur II_1, muni d'une trace fidèle donnée par $\tau(x) = \langle x\delta_e, \delta_e \rangle$, $x \in L\Gamma$.

- Soit $\Gamma \curvearrowright (X, \mu)$ une action non-singulière sur un espace sans atomes. L'algèbre $M := L^\infty(X, \mu) \rtimes \Gamma$ est un facteur pourvu que l'action soit libre et ergodique. Ce facteur est de type II s'il existe une mesure ν, finie (cas II_1) ou non (cas II_∞), équivalente à μ qui est Γ-invariante. M est de type III sinon.

I.1.3 Historique sur les facteurs II_1

Naturellement, les facteurs de dimension infinie les plus simples à étudier sont à priori ceux qui sont "approchables" par des sous-algèbres de dimension finie. C'est ainsi qu'est née la notion d'algèbre de von Neumann *hyperfinie* : M est dite hyperfinie s'il existe une suite croissante de sous-$*$-algèbres de dimension finie $M_n \subset M$ telles que $M = (\cup_n M_n)''$.

Dans [MvN36], Murray et von Neumann montrent que tous les facteurs hyperfinis de type II_1 sont isomorphes. Pour cette raison, on parle *du* facteur II_1 hyperfini, souvent noté R.

Une question évidente s'impose alors à eux : est-ce que tout facteur II_1 est isomorphe à R, ou bien y a t'il des facteurs II_1 non-hyperfinis ?

Ils répondent à cette question en montrant que le facteur du groupe libre à deux générateurs LF_2 n'est pas hyperfini. Plus précisément, ils prouvent que LF_2 n'a pas de *suites centrales* non-triviales. Sans entrer dans les détails, cela provient du fait que F_2 n'est pas intérieurement moyennable. À contrario, R possède des suites centrales non-triviales.

Cependant, Murray et von Neumann ne parviennent pas à donner d'autres exemples de facteurs non hyperfinis. Notamment, ils posent la question suivante, encore largement ouverte aujourd'hui.

Question I.1.5 (Free group factors problem)**.** *Si deux groupes libres sont non-isomorphes, est-ce le cas de leurs algèbres de von Neumann ?*

Une autre question très naturelle qu'ils laissent ouverte : est-ce qu'un sous-facteur de R est hyperfini ? Si oui, alors le facteur $R \overline{\otimes} LF_2$ n'est pas hyperfini mais contient tout de même des suites centrales non-triviales.

Il aura fallu attendre près de 30 ans pour que McDuff ([McD69a, McD69b]) démontre l'existence d'une infinité de facteurs II_1 deux à deux non-isomorphes. Ses travaux reposent sur des arguments de suites centrales déjà présents dans les travaux de Murray et von Neumann. Avec une compréhension plus grande de ces suites centrales, McDuff [McD70] donne également une caractérisation des facteurs de la forme $R \overline{\otimes} M$, appelés par la suite *facteurs McDuff*.

En 1976, une nouvelle avancée majeure survient avec les travaux de Connes [Co76], qui montre que tout facteur II_1 moyennable est hyperfini et donc isomorphe à R. La moyennabilité pour les algèbres de von Neumann est définie de manière analogue à la moyennabilité des groupes. En plus d'offrir une description précise des facteurs II_1 moyennables, le résultat de Connes permet de répondre à la question de Murray et von Neumann : une sous-algèbre de von Neumann du facteur II_1 hyperfini est elle-même hyperfinie.

Ce résultat témoigne d'une grande perte d'information dans le passage à l'algèbre de von Neumann $L\Gamma$ d'un groupe Γ. Plus tard, Connes [Co80] observe à l'inverse certaines propriétés de rigidité vis-à-vis de cette construction pour des groupes avec la propriété (T) de Kazhdan. Il émet alors la conjecture suivante.

Conjecture I.1.6 (Conjecture de rigidité de Connes)**.** *Si Γ est un groupe ICC avec la propriété (T) alors tout groupe Λ tel que $L\Gamma \simeq L\Lambda$ est isomorphe à Γ.*

Tout comme la question I.1.5, cette conjecture est largement ouverte. De manière générale, on ne connait à ce stade que très peu de choses sur les facteurs II_1 non-moyennables.

Deux axes de recherche vont alors se dessiner :

- la théorie des sous-facteurs d'une part, initiée par Jones, consistant à étudier une inclusion de deux facteurs avec des outils combinatoires tels que l'invariant standard, ou les algèbres planaires ;
- la théorie des probabilités libres d'autre part, introduite par Voiculescu dans le but de comprendre plus en profondeur les facteurs de groupes libres LF_n. Sans résoudre le problème

d'isomorphisme I.1.5, cette théorie a permis tout de même de démontrer [Vo96] que les facteurs LF_n ne possédaient pas de sous-algèbre de Cartan (voir la Section I.2.1).

En parallèle, la recherche se développe dans le domaine des relations d'équivalence sur des espaces mesurés. Ce développement va induire un troisième axe de recherche : la théorie de la déformation/rigidité élaborée par Popa, dont le but premier était de démontrer des propriétés de rigidité de la group measure space construction (Exemple I.1.3).

Mon travail de thèse s'inscrit dans cette théorie. Dans les Sections I.2 et I.3, je décris le lien profond entre les algèbres de von Neumann et les relations d'équivalence et j'explique les grandes idées et avancées de la déformation/rigidité. Mes travaux personnels seront présentés dans la Section I.4.

I.2 Relations d'équivalence mesurées et algèbres de von Neumann

I.2.1 Relations d'équivalence et sous-algèbres de Cartan

L'exemple I.1.3 explique comment construire une algèbre de von Neumann de type produit croisé $M := L^\infty(X,\mu) \rtimes \Gamma$ à partir d'une action non singulière $\Gamma \curvearrowright (X,\mu)$ sur un espace de probabilité sans atome. Cette algèbre est un facteur II_1 dès lors que l'action est libre, ergodique et *pmp* (*i.e.* préserve la mesure de probabilité μ). Dans ce cas, la sous-algèbre $A := L^\infty(X,\mu) \subset M$ est une *sous-algèbre de Cartan* :

- elle est *maximale abélienne* au sens où $A' \cap M = A$;
- elle est *régulière*, *i.e.* le normalisateur $\{u \in \mathcal{U}(M)\,,\, uAu^* = A\}$ engendre M comme algèbre de von Neumann. Ici $\mathcal{U}(M)$ désigne l'ensemble des *unitaires* de M, *i.e.* les éléments $u \in M$ tels que $uu^* = u^*u = 1$.

La paire $A \subset M$ ne dépend, à isomorphisme près, que de la relation d'équivalence $\mathcal{R}_\Gamma$ sur X donnée par les Γ-orbites.

Réciproquement, Feldman et Moore [FM77] montrent que cette relation d'équivalence $\mathcal{R}_\Gamma$ est un invariant d'isomorphisme de la paire $A \subset M$. Autrement dit, pour deux actions $\Gamma \curvearrowright (X,\mu)$ et $\Lambda \curvearrowright (Y,\nu)$, les relations d'équivalence $\mathcal{R}_\Gamma$ et $\mathcal{R}_\Lambda$ sont isomorphes si et seulement si on a un isomorphisme de paires

$$(L^\infty(X,\mu) \subset L^\infty(X,\mu) \rtimes \Gamma) \simeq (L^\infty(Y,\nu) \subset L^\infty(Y,\nu) \rtimes \Lambda).$$

Plus généralement, ils montrent que toute paire $A \subset M$ où M est un facteur et A une sous algèbre de Cartan, est associée à une unique relation d'équivalence mesurée $\mathcal{R}$ sur un espace (X,μ) par une construction similaire.

Leurs travaux établissent un lien fort entre algèbres de von Neumann et relations d'équivalence mesurées non sans conséquence sur les avancées dans les deux domaines. Notamment, on observe un historique sur les relations d'équivalence hyperfinies et moyennables [Dy59, OW80, CFW81] similaire à ce qui a été exposé pour les algèbres de von Neumann en I.1.3.

I.2.2 W*-(super)rigidité

Toutes les actions considérées dans la suite de cette introduction seront supposées libres ergodiques. Nous écrirons pmp lorsqu'elles préservent une mesure de probabilité.

Définition I.2.1. Deux actions pmp $\Gamma \curvearrowright (X,\mu)$ et $\Lambda \curvearrowright (Y,\nu)$ sont dites :

- *Conjuguées* s'il existe un isomorphisme de groupes $\phi : \Gamma \to \Lambda$ et un isomorphisme d'espaces mesurés (bimesurable préservant la mesure) $\Delta : X \to Y$ tels que $\Delta(s \cdot x) = \phi(s) \cdot \Delta(x)$, pour presque tout $x \in X$ et tout $s \in \Gamma$.
- *Orbitalement équivalentes* (OE) s'il existe un isomorphisme d'espaces mesurés (bimesurable préservant la mesure) $\Delta : X \to Y$ tel que $\Delta(\Gamma \cdot x) = \Lambda \cdot \Delta(x)$ pour presque tout $x \in X$. De manière équivalente, elles sont OE si on a un isomorphisme de paires

$$(L^\infty(X,\mu) \subset L^\infty(X,\mu) \rtimes \Gamma) \simeq (L^\infty(Y,\nu) \subset L^\infty(Y,\nu) \rtimes \Lambda).$$

- *W^*-équivalents* si les produits croisés par ces actions sont isomorphes :

$$L^\infty(X,\mu) \rtimes \Gamma \simeq L^\infty(Y,\nu) \rtimes \Lambda.$$

Clairement, on a les implications

$$\text{conjugaison} \Rightarrow \text{équivalence orbitale} \Rightarrow W^*\text{-équivalence}.$$

La question des implications réciproques est à l'origine de la théorie de la déformation/rigidité de Popa. Le point de départ qui a vraiment motivé les travaux de Popa est certainement le résultat de Gaboriau sur les actions des groupes libres.

Théorème I.2.2 (Gaboriau, [Ga00])**.** *Des actions pmp de groupes libres de rangs différents ne sont jamais orbitalement équivalentes.*

Ce résultat amène la question suivante, qui rappelle fortement le problème de Murray et von Neumann I.1.5.

Question I.2.3. *Si $n \neq m$, existe-t-il des actions pmp $F_n \curvearrowright (X,\mu)$ et $F_m \curvearrowright (Y,\nu)$ telles que*

$$L^\infty(X,\mu) \rtimes F_n \simeq L^\infty(Y,\nu) \rtimes F_m\,?$$

De même, on peut envisager une version dynamique de la conjecture de Connes I.1.6. Nous devons cependant restreindre la classe d'actions considérées pour espérer conclure un isomorphisme au niveau des groupes.

Question I.2.4. *Soient Γ et Λ deux groupes avec la propriété (T), et $\Gamma \curvearrowright (X,\mu)$ et $\Lambda \curvearrowright (Y,\nu)$ deux actions pmp, disons de Bernoulli, W*-équivalentes. Est-ce que les groupes Γ et Λ sont isomorphes ? Est-ce que les actions sont conjuguées ? Et si l'on suppose seulement que Γ a la propriété (T) ?*

La version la plus générale de cette question demande en fait si toute action de Bernoulli d'un groupe ICC avec la propriété (T) est W*-superrigide, au sens suivant.

Définition I.2.5. Une action pmp σ est *OE-superrigide* (resp. *W*-superrigide*) si toute action orbitalement équivalente (resp. W*-équivalente) à σ est en fait conjuguée à σ.

Existe-t-il des actions OE/W*-superrigides ? Notons que pour prouver qu'une action est W*-superrigide on peut procéder en deux étapes : d'abord on montre qu'elle est OE-superrigide, puis on montre que l'algèbre de von Neumann associée a une unique sous-algèbre de Cartan. Chacune de ces deux étapes constitue un problème très difficile. La déformation/rigidité permet de résoudre ces deux problèmes, dans des situations variées.

I.3 Théorie de la déformation/rigidité

L'idée de la théorie de la déformation/rigidité de Popa est de confronter certaines propriétés de rigidité d'un facteur II_1 M à une *déformation* de ce facteur, c'est à dire un groupe d'automorphismes à un paramètre $(\alpha_t)_{t\in\mathbb{R}}$ tel que $t \mapsto \alpha_t(x)$ est faiblement continu pour tout $x \in M$.

Par exemple si une sous-algèbre $Q \subset M$ a la propriété (T)[1], alors toute déformation (α_t) converge *uniformément* vers l'identité sur la boule unité (pour la norme d'opérateurs) de Q. Ainsi on peut espérer identifier Q à la sous-algèbre des points fixes de (α_t).

Pour plus de détails sur la mise en pratique de ces principes de déformation et rigidité, nous renvoyons à la Section II.2.

I.3.1 W*-(super)rigidité : quelques résultats

Comme nous l'avons vu dans la Section I.1.3, les algèbres de von Neumann sont difficiles à comprendre. Mais dans certains contextes la déformation/rigidité va s'avérer particulièrement efficace pour prouver des résultats de W*-rigidité et enfin commencer à comprendre le cas non-moyennable.

Par exemple, les actions par décalage de Bernoulli sont des actions avec de fortes propriétés de déformation. C'est cette observation qui a permis à Popa [Po06a, Po06b] de montrer non-seulement que l'action de Bernoulli d'un groupe ICC avec la propriété (T) est OE-superrigide, mais aussi de démontrer le premier résultat de W*-rigidité : si Γ et Λ sont ICC et ont la propriété (T) et si leurs actions de Bernoulli sont W*-équivalentes alors les groupes sont isomorphes et les actions sont conjuguées.

Plus récemment, Ioana [Io11] a montré la W*-superrigidité des actions de Bernoulli des groupes ICC avec la propriété (T). C'est l'un des premiers exemples d'actions W*-superrigides (voir aussi [Pe(09), PV10a] et plus récemment [IPV13, HPV13, CIK(13)]). Ce résultat répond donc partiellement à la question I.2.4. Dans la même direction, j'ai prouvé dans [Bo13] la W*-superrigidité de toutes les actions Gaussiennes mélangeantes de groupes ICC avec la propriété (T). Voir la Section II.1.1 pour la définition des actions Gaussiennes.

Conjecturalement, les résultats de Popa et Ioana [Po06a, Po06b, Io11] se généralisent au cadre des groupes non-moyennables généraux.

Conjecture I.3.1. *Le facteur associé à l'action de Bernoulli de tout groupe non-moyennable a une unique sous-algèbre de Cartan, à conjugaison près.*

[1] Un analogue de la propriété (T) pour les groupes.

Précisons que la conjecture n'est même pas connue pour les groupes avec la propriété (T), contrairement à ce que pourrait laisser penser le résultat de Ioana [Io11] mentionné ci-dessus.

Voici une autre conjecture en lien étroit avec le problème I.2.3. Cette conjecture laisse entrevoir la possibilité de définir une cohomologie intéressante pour les facteurs II_1.

Conjecture I.3.2. *Le premier nombre de Betti ℓ^2 d'un groupe Γ est un invariant d'isomorphisme de tout facteur $L^\infty(X,\mu) \rtimes \Gamma$ associé à une action pmp $\Gamma \curvearrowright (X,\mu)$.*

On sait que ce nombre de Betti $\beta_1^{(2)}(\Gamma) \geq 0$ est un invariant de la relation d'équivalence orbitale $\mathcal{R}_\Gamma$ associée à toute action de Γ ([Ga02]). Donc pour résoudre la conjecture, il suffit de montrer que pour tout groupe Γ avec un nombre de Betti non-nul et toute action pmp de Γ, le facteur associé a une unique sous-algèbre de Cartan.

Dans le cadre des groupes libres, le premier résultat est dû à Ozawa et Popa [OP10a]. Ils montrent que toute action *profinie* de F_n produit un facteur II_1 qui a une unique sous-algèbre de Cartan, à conjugaison unitaire près. De nombreuses généralisations de ce résultat ont alors suivi [OP10b, CS13, CSU13, PV(12), PV(13), Io(12)a, CIK(13)]. Notamment Popa et Vaes ont montré que pour **toute** action pmp $\Gamma \curvearrowright (X,\mu)$ d'un groupe hyperbolique, le produit croisé $L^\infty(X,\mu) \rtimes \Gamma$ a une unique sous-algèbre de Cartan à conjugaison unitaire près.

Ils répondent en particulier à la question I.2.3. Cependant la conjecture I.3.2 n'est pas encore totalement résolue.

Cette méthode d'unicité des sous-algèbres de Cartan n'a pas d'analogue dans le cas des algèbres de groupes $L\Gamma$, et les problèmes I.1.5 et I.1.6 demeurent entièrement ouverts. Cependant on sait aujourd'hui [IPV13, BV(13)] qu'il existe des groupes W*-superrigides (*i.e.* tels que Γ est un invariant d'isomorphisme de $L\Gamma$).

I.3.2 Propriétés structurelles des facteurs II_1

Outre les résultats de rigidité mentionnés ci-dessus, la déformation/rigidité a permis de nombreuses avancées dans la compréhension des facteurs II_1. Notamment on a observé de grands progrès sur le calculs d'invariants tels que le *groupe fondamental* ou le groupe d'*automorphismes extérieurs*, [Po06c, Po06a, IPP08, Ho09, PV10b, De10].

D'autre part, la théorie de déformation/rigidité a révélé de nombreuses propriétés structurelles de certains facteurs II_1. Voici une liste non exhaustive de telles propriétés structurelles.

- Absence ou unicité des sous-algèbres de Cartan. Nous avons déjà discuté ce point.
- *Primalité*. Un facteur II_1 M est premier si on ne peut pas l'écrire comme produit tensoriel de deux facteurs II_1.
- *Solidité* ([Oz04]). M est solide si pour toute algèbre diffuse[2] $Q \subset M$ le commutant relatif $Q' \cap M$ est moyennable.
- *Solidité forte* ([OP10a]). M est fortement solide si pour toute sous-algèbre diffuse moyennable $Q \subset M$, le normalisateur $\mathcal{N}_M(Q) := \{u \in \mathcal{U}(M) \,|\, uQu^* = Q\}$ engendre une algèbre de von Neumann moyennable.

[2] Une algèbre de von Neumann Q est dite *diffuse* lorsqu'elle n'a pas de projection p minimale, *i.e.* telle que $pQp = \mathbb{C}p$.

Noter que si M n'est pas moyennable on a des implications

$$M \text{ fortement solide} \Rightarrow M \text{ solide} \Rightarrow M \text{ premier},$$
$$M \text{ fortement solide} \Rightarrow M \text{ n'a pas de sous-algèbre de Cartan.}$$

Les premiers facteurs étudiés dans le cadre de ces propriétés sont les facteurs des groupes libres LF_n. Comme nous l'avons dit plus haut, Voiculescu [Vo96] a montré avec des techniques de probabilités libres que ces facteurs n'avaient pas de sous-algèbre de Cartan. Par des techniques nouvelles, issues des C*-algèbres, Ozawa [Oz04] a prouvé qu'ils étaient solides. Ce sont Ozawa et Popa [OP10a] qui ont démontré qu'ils étaient en fait fortement solides. Ensuite de nombreuses généralisations ont suivi pour différentes classes de facteurs [Ho10, HS11, CS13, CSU13, Io(12)a, Ho(12)b, Va(13)].

Mentionnons que ces propriétés ont des analogues pour des facteurs de type III, et la plupart des résultats précédents admettent des généralisations [CH10, HR11, Is(12), BHR14]. Notamment, Houdayer et Vaes [HV13] ont démontré des résultats d'unicité de sous-algèbres de Cartan dans les facteurs associés à certaines actions non-singulières des groupes libres (et plus généralement des groupes hyperboliques). Vaes [Va(13)] a exhibé le premier exemple d'action non-singulière (de type III_1) qui soit W*-superrigide.

I.4 Principaux résultats, contenu des chapitres

Dans cette thèse je m'intéresse à divers aspects de rigidité des algèbres de von Neumann dans trois contextes différents :

- les algèbres associées à des actions Gaussiennes ;
- les produits libres amalgamés d'algèbres de von Neumann ;
- les algèbres de groupes.

Chacun de ces trois cas correspond à un chapitre de ma thèse.

I.4.1 Chapitre II : Crossed-product von Neumann algebras associated with Gaussian actions

Ce chapitre est la fusion de deux articles [Bo12, Bo13]. J'y étudie les propriétés des actions Gaussiennes et des algèbres de von Neumann correspondantes.

Les actions Gaussiennes sont des actions pmp fonctoriellement associées à des représentations de groupes et généralisent la notion d'action de Bernoulli (voir la Section II.1.1 pour une définition plus précise).

Le but est de généraliser les principaux résultats sur les actions de Bernoulli obtenus grâce à la déformation rigidité [Po06a, Po06b, Po08, Io11, IPV13] : résultats de W*-superrigidité, calculs d'invariants, propriétés structurelles des produits croisés associés à ces actions. La principale difficulté dans ce travail tient au fait que les actions Bernoulli ont une structure algébrique riche et de très fortes propriétés de mélange, ce qui n'est pas le cas des actions Gaussiennes générales.

Par rapport aux articles [Bo12, Bo13], certains résultats ont été améliorés et de nouvelles observations viennent compléter le sujet. La présentation est aussi rendue plus accessible, en ne montrant les résultats que dans des cas simples (et en me référant aux articles pour les cas plus généraux).

Voici deux théorèmes qui synthétisent les résultats que je démontre dans le Chapitre II.

Théorème I.4.1 (Voir Theorem II.2.20)**.** *Soit* $\Gamma \curvearrowright (X, \mu)$ *l'action Gaussienne associée à une représentation mélangeante faiblement contenue dans la régulière. Notons* $M = L^\infty(X, \mu) \rtimes \Gamma$*.*

Pour toute sous-algèbre $Q \subset M$ *contenant* $L^\infty(X, \mu)$*, il existe des projections* $(p_n)_{n\in\mathbb{N}}$ *dans le centre de* Q *telles que* p_0Q *est hyperfinie, et pour tout* $n \geq 1$, p_nQ *est un facteur premier sans suite centrale non-triviale.*

En particulier la relation d'équivalence orbitale $\mathcal{R}_\Gamma$ associée à une action Gaussienne $\Gamma \curvearrowright (X, \mu)$ comme dans le théorème est *solidement ergodique* : toute sous-relation d'équivalence $\mathcal{R} \subset R_\Gamma$ se décompose en une partie hyperfinie et un nombre dénombrable de sous-ensembles invariants sur lesquels $\mathcal{R}$ est fortement ergodique.

Le théorème I.4.1 donne donc des informations sur les relations d'équivalence à partir de leurs algèbres de von Neumann. Ce résultat avait été démontré pour les actions de Bernoulli par Chifan et Ioana [CI10].

Dans la direction de la rigidité des groupes avec la propriété (T) (Question I.2.4), je montre le résultat suivant, généralisant [Io11].

Théorème I.4.2 (Voir Theorem II.4.5 et II.4.6)**.** *Soient* Γ *un groupe ICC avec la propriété (T) et* π *une représentation mélangeante de* Γ*. Notons* $\sigma_\pi : \Gamma \curvearrowright (X, \mu)$ *l'action Gaussienne associée à* π*. Les propriétés suivantes sont satisfaites.*

1. *L'action* σ_π *est* W^**-superrigide.*
2. *L'algèbre* $M := L^\infty(X, \mu) \rtimes \Gamma$ *a un groupe fondamental trivial : la seule projection* p *telle que l'algèbre réduite* pMp *soit isomorphe à* M *est la projection identité* $p = 1$*.*
3. *Si* π *n'est pas faiblement contenue dans la régulière, aucune amplification* $p(M \otimes M_n(\mathbb{C}))p$ *n'est isomorphe à une algèbre de groupe* $L\Lambda$*.*

I.4.2 Chapitre III : Amalgamated free product type III factors with at most one Cartan subalgebra

Ce chapitre est le fruit d'une collaboration avec Cyril Houdayer et Sven Raum, et a été publié dans Compositio Mathematica. Nous étudions les sous-algèbres de Cartan dans des produits libres amalgamés d'algèbres de von Neumann générales.

Ioana [Io(12)a] a étudié cette question dans le cas des algèbres de von Neumann traciales. Il a ainsi montré que le produit libre tracial non-moyennable de deux algèbres de von Neumann ne contient jamais de sous-algèbre de Cartan. Il obtient aussi des résultats dans le cas amalgamé. Par exemple, si $\Gamma = \Gamma_1 * \Gamma_2$ avec $|\Gamma_1| \geq 2$ et $|\Gamma_2| \geq 3$, alors le facteur associé à toute action libre ergodique, pmp $\Gamma \curvearrowright (X, \mu)$ admet une unique sous-algèbre de Cartan, à conjugaison unitaire près.

Nous étendons ces résultats aux algèbres de von Neumann quelconques en utilisant les travaux de Connes et Takesaki [Co73, CT77, Ta03]. Dans le cas où l'amalgame est trivial, nous obtenons le résultat optimal suivant.

Théorème I.4.3 (Voir Theorem III.A)**.** *Si (M_1, φ_1) et (M_2, φ_2) sont des algèbres de von Neumann (agissant sur des espaces de Hilbert séparables) telles que* $\dim(M_1) \geq 2$ *and* $\dim(M_2) \geq 3$ *alors le produit libre $(M_1, \varphi_1) * (M_2, \varphi_2)$ n'a pas de sous-algèbre de Cartan.*

Ce théorème signifie que les produits libres ne sont jamais associés à des relations d'équivalence [FM77]. Dans le cas des produits libres amalgamés au dessus de sous-algèbres moyennables, nous obtenons également des résultats structurels dont nous déduisons le résultat suivant.

Théorème I.4.4 (Voir Theorem III.C)**.** *Soit $\mathcal{R}$ une relation d'équivalence ergodique non-singulière sur un espace de probabilité standard (X, μ). Supposons que $\mathcal{R}$ se décompose comme un produit libre $\mathcal{R} = \mathcal{R}_1 * \mathcal{R}_2$ de deux relations d'équivalence récurrentes.*

Alors l'algèbre de von Neumann associée $L\mathcal{R}$ ([FM77]) admet $L^\infty(X, \mu)$ comme unique sous-algèbre de Cartan, à conjugaison unitaire près.

La définition de décomposition en produit libre d'une relation d'équivalence est rappelée au Chapitre III. Une relation $\mathcal{R}$ est dite *récurrente* si elle satisfait la conclusion du théorème de récurrence de Poincaré : pour tout ensemble $\mathcal{U} \subset X$ de mesure positive, pour presque tout $x \in \mathcal{U}$, la $\mathcal{R}$-classe d'équivalence de x contient une infinité de points dans $\mathcal{U}$.

Pour illustrer ce théorème, prenons un groupe $\Gamma = \Gamma_1 * \Gamma_2$, et une action libre ergodique non-singulière $\Gamma \curvearrowright (X, \mu)$ telles que pour $i = 1, 2$, la relation $\mathcal{R}_{\Gamma_i}$ est récurrente. Alors le produit croisé $L^\infty(X, \mu) \rtimes \Gamma$ a une unique sous-algèbre de Cartan. Nous traitons aussi le cas où Γ est un produit libre amalgamé, avec un amalgame fini (Voir Theorem III.D).

Par rapport au cas où l'action préserve la mesure, nous ajoutons l'hypothèse de récurrence de chacun des facteurs libres. Nous montrons que cette hypothèse est nécessaire en construisant un exemple où elle n'est pas satisfaite et où le facteur a une infinité de sous-algèbres de Cartan deux à deux non conjuguées.

I.4.3 Chapitre IV : Maximal amenable subalgebras of von Neumann algebras associated with hyperbolic groups

Contrairement aux deux autres chapitres, les travaux présentés dans ce chapitre, effectués en collaboration avec Alessandro Carderi, ne reposent pas sur la théorie de déformation/rigidité. Notre approche consiste à utiliser la géométrie des groupes hyperboliques pour récupérer des informations sur les algèbres de von Neumann qui leur sont associées.

Notre objectif est d'étudier les sous-algèbres maximales moyennables dans les facteurs de groupes hyperboliques. La motivation première de ce travail (et des travaux analogues [Po83, CFRW10, Ho(12)a]) est une question de Kadison, posée dans les années 1960. Murray et von Neumann avaient remarqué que tout facteur II_1 contenait le facteur hyperfini. Kadison s'interrogea sur la fréquence de ce phénomène : y a t'il beaucoup de sous-facteurs hyperfinis dans les facteurs II_1 ? Plus formellement voici la question posée.

Question I.4.5 (Kadison)**.** *Dans un facteur II_1, est-ce que tout élément est contenu dans un facteur hyperfini ?*

Popa [Po83] a répondu à cette question par la négative : il a montré que la sous-algèbre de LF_2 engendrée par l'un des générateurs a et b de F_2 est maximale moyennable. Donc aucun générateur n'est contenu dans un facteur hyperfini. Plus récemment, les auteurs de [CFRW10] ont montré que a sous-algèbre radiale (engendrée par $a + a^{-1} + b + b^{-1}$) est aussi maximale moyennable.

Nous donnons beaucoup plus d'exemples de ce phénomène.

Théorème I.4.6 (Voir Theorem IV.A)**.** *Si* Γ *est un groupe hyperbolique et* $\Lambda < \Gamma$ *un sous-groupe qui est maximal moyennable alors* $L\Lambda \subset L\Gamma$ *est maximale moyennable.*

Comme dans un groupe libre F_n tout élément qui n'est pas une puissance engendre un sous groupe maximal moyennable, l'élément correspondant dans LF_n n'est pas contenu dans un facteur hyperfini. En fait nos techniques montrent que si Γ est un groupe hyperbolique, aucun élément d'ordre infini de Γ n'est contenu dans un facteur hyperfini.

Nous généralisons ce résultat dans plusieurs directions. Par exemple, nous traitons aussi le cas des groupes relativement hyperboliques, des produits de tels groupes, et également des produits croisés associés à des actions sur des algèbres moyennables.

Ainsi nous montrons que pour toute action libre pmp $\Gamma \curvearrowright (X, \mu)$ d'un groupe hyperbolique, et pour tout sous groupe $\Lambda < \Gamma$ maximal moyennable, la relation d'équivalence $\mathcal{R}_\Lambda \subset \mathcal{R}_\Gamma$ est maximale hyperfinie.

I.4.4 Appendice : Mixing bimodules over finite von Neumann algebras

Dans cet appendice, je présente de manière unifiée plusieurs aspects de mélange des algèbres de von Neumann. Je développe la notion de *bimodule mélangeant*, introduite par Peterson. Un bimodule est l'analogue en algèbres de von Neumann de la notion de représentation de groupes.

Cette notion permet notamment d'apporter un nouvel éclairage sur le fameux théorème d'entrelacement de Popa, grâce à l'observation suivante.

Pour deux sous-algèbres $A, B \subset M$ d'un facteur II_1, A se plonge dans B au sens de Popa si et seulement si le bimodule ${}_A L^2(M)_B$ n'est pas faiblement mélangeant. Le théorème de Popa découle alors de la multiplicité des caractérisations de la notion de mélange faible d'un bimodule.

Notations

General notations

On $B(H)$, we will denote the operator norm by $\|\cdot\|$ or $\|\cdot\|_\infty$, depending on the context.

If $M \subset B(H)$ is a von Neumann algebra and Q is a subalgebra, we use the following notations.

- $(M)_1$ is the unit ball of M for the norm $\|\cdot\|$;
- $\mathcal{U}(M) = \{u \in M \,|\, uu^* = u^*u = 1\}$ is the unitary group of M;
- $\mathcal{N}_M(Q) = \{u \in \mathcal{U}(M) \,|\, uQu^* = Q\}$ is the normalizer of Q inside M;
- $\mathcal{QN}_M(Q)$ is the quasi-normalizer of Q. It is the set of elements $x \in M$ for which there exist finitely many $y_1, \cdots, y_k \in M$ such that

$$xQ \subset \sum_{i=1}^{k} Qy_i \text{ and } Qx \subset \sum_{i=1}^{k} y_iQ.$$

When we consider a finite von Neumann algebra M, we always denote by τ a faithful normal trace on M. If M arises from a specific construction (*e.g.* M comes from a group, or an equivalence relation) then we choose τ to be the canonical trace coming from this construction.

For $p = 1, 2$, we will denote by $\|\cdot\|_p$ the p-norm on M associated with τ : $\|x\|_p = \tau(|x|^p)^{1/p}$, for all $x \in M$. $L^2(M)$ will be the GNS construction with respect to τ.

Whenever $Q \subset M$ is a subalgebra, $E_Q : M \to Q$ denotes the unique trace preserving conditional expectation onto Q.

If ω is a free ultrafilter on $\mathbb{N}$, we denote by M^ω the corresponding ultraproduct von Neumann algebra. It is the set of $\|\cdot\|$-bounded sequence, up to the following identification:

$$(x_n) \sim (y_n) \text{ if and only if } \lim_{n\to\omega} \|x_n - y_n\|_2 = 0.$$

It is itself a von Neumann algebra.

The symbol $A \prec_M B$ means that a corner of A embeds into B inside M in the sense of Popa. See Section A.3 for the definition (and Section III.2.1 in the case where M is not finite).

For any $t > 0$, we denote by M^t the t-amplification of M defined (up to isomorphism) by choosing an integer $n \geq t$ and a projection $p \in M \otimes M_n(\mathbb{C})$ with trace t, and by setting $M^t = p(M \otimes M_n(\mathbb{C}))p$. The fundamental group of M is given by

$$\mathcal{F}(M) = \{t \in \mathbb{R}_+^* \,|\, M \simeq M^t\}.$$

Finally, we will adopt the following convention. Except for ultraproduct algebras, all von Neumann algebras that we consider will be assumed to have separable predual.

Chapter II

Crossed-product von Neumann algebras associated with Gaussian actions

In this chapter we study Gaussian actions at the level of ergodic theory and von Neumann algebras. We thus investigate properties such as strong ergodicity, solid ergodicity ([Ga10, Definition 5.4]) at the ergodic theoretic level, while we study W*-rigidity phenomena at the von Neumann algebraic level.

Contents

In the context of von Neumann algebras, Gaussian actions first appeared in [Po06a, Fu07, Po08] as examples of malleable actions, for which orbit equivalence superrigidity results have been proven. Also they have been used by Peterson and Sinclair [PS12] for their nice behaviour with respect to group cohomology. This idea in [PS12] that the Gaussian construction allows to construct a deformation of $L\Gamma$ out of a 1-cocycle of Γ, was further developed in [Si11], [Va13], and [CS13] allowing to prove very general strong solidity results, and uniqueness of Cartan subalgebras results.

II.1 Gaussian actions

Gaussian actions are measure-preserving actions which are constructed from group representations. Historically they first appeared in probability theory [Ne71, CFS82], and they were called *random Gaussian processes*. However they are closely related to Gaussian Hilbert spaces and Wiener Chaos decomposition, going back to the work of Segal [Se56], and mathematical physics. We refer to [Ja97] for more on this.

II.1.1 Several descriptions of Gaussian actions.

From the multiple constructions of Gaussian Hilbert spaces, one can deduce several ways of defining Gaussian actions. In order to give a complete picture of Gaussian actions we will explain the main historical constructions.

Classically [CFS82, Ja97, Gl03] Gaussian actions are constructed by the means of a covariance matrix (as in Section II.1.1.3). A construction with creation operators on a symmetric Fock space is also given in [PS12], which will be explained in Section II.1.1.2. Another definition is given in [BHV08], but we will not present it.

The link between these descriptions is the so-called Wiener chaos decomposition (see [Ja97]). However, Vaes [Va13] found an easier way to check that these definitions coincide; he gave an abstract characterization of Gaussian actions. This characterization will be our main definition.

In the three paragraphs below, the initial data is an orthogonal representation $\pi : \Gamma \to \mathcal{O}(H)$ on a real Hilbert space and the aim is to construct a measure-preserving action σ_π of Γ on a standard probability space (X_π, μ). Equivalently, we want to construct a trace preserving action σ_π of Γ on an abelian tracial von Neumann algebra (A_π, τ).

II.1.1.1 Finite dimensional approach and universal description

In the case where the representation $\pi : \Gamma \to \mathcal{O}(H)$ is finite dimensional, the construction of σ_π is extremely simple:

- Pick an orthonormal basis $(e_i)_{i=1,\cdots,n}$ of H, so that $H \simeq \mathbb{R}^n$. Now endow $H \simeq \mathbb{R}^n$ with the product measure $\mu = \nu^{\otimes n}$ of the standard Gaussian measure $\nu \in \text{Prob}(\mathbb{R})$.

- The measure μ does not depend on the choice of the orthonormal basis $(e_i)_i$ and (equivalently) it is invariant under the action of $\mathcal{O}(H)$.

- So π induces a measure preserving action σ_π on the canonical probability space $(X_\pi, \mu) := (H, \mu)$. This is the Gaussian action associated to π.

In our applications we will be mostly interested in infinite dimensional representations. Naively, one might want to proceed as before. The main problem is that if we identify H with $\ell^2(\mathbb{N})$ by fixing an orthonormal basis, the subset $H \subset (R^{\mathbb{N}}, \nu^{\otimes \mathbb{N}})$ has measure 0.

So let us first analyse the finite dimensional case in more details. The first question to ask: how can one compare the initial representation π to the resulting Gaussian action σ_π?

Consider the Koopman representation, again denoted by $\sigma_\pi : \Gamma \to \mathcal{O}(L^2_{\mathbb{R}}(X_\pi, \mu))$. Then π appears as a subrepresentation of σ_π. Indeed, any vector $\xi \in H$, gives rise to a function $f_\xi : \eta \mapsto \langle \eta, \xi \rangle$ on $X_\pi (= H)$ in such a way that $\sigma_{\pi,g}(f_\xi) = f_{\pi(g)\xi}$ for all $g \in \Gamma$. The functions f_ξ are centered Gaussian random variables.

Now one can check that the algebra generated by the functions f_ξ, $\xi \in H$ remains inside $L^2_{\mathbb{R}}(X_\pi, \mu)$ and forms a dense subspace. Hence we see that the Gaussian action is completely determined by the restriction of its Koopman representation to some *generating Gaussian Hilbert space.*

Definition II.1.1. A *Gaussian Hilbert space* K is a closed subspace of $L^2_{\mathbb{R}}(X, \mathcal{B}, \mu)$ for some standard probability space $(X, \mathcal{B}, \mu)$, such that every element of K is a centered Gaussian random variable.

K is said to be *generating* if the σ-subalgebra of $\mathcal{B}$ generated by all the random variables $X \in K$ is $\mathcal{B}$ itself.

Let us use these Gaussian Hilbert spaces to define Gaussian actions associated with infinite dimensional representations.

Proposition II.1.2. *Any real Hilbert space H is isomorphic to a generating Gaussian Hilbert space $K \subset L^2_{\mathbb{R}}(X, \mathcal{B}, \mu)$*

Proof. Take an orthonormal basis $(e_i)_{i \in I}$ of H and identify $H \simeq \ell^2(I)$. Consider the probability space $(\mathbb{R}^I, \nu^{\otimes I})$, where ν is the standard Gaussian measure. For any $i \in I$, the projection $P_i : (x_j) \in \mathbb{R}^I \mapsto x_i$ defines a standard Gaussian random variable. Then $K := \overline{\text{span}}\{P_i\, , i \in I\}$ is a Gaussian Hilbert space ([Ja97, Theorem 1.3]), which is clearly generating and the map $e_i \mapsto P_i$ gives rise to an isomorphism from H onto K. $\square$

Proposition II.1.3. *If $K \subset L^2_{\mathbb{R}}(X, \mathcal{B}, \mu)$ is a generating Gaussian Hilbert space then the unitaries $w(\xi) := \exp(i\sqrt{2}\xi) \in L^\infty(X, \mu)$, $\xi \in K$ satisfy the following properties, where τ denotes the state on $L^\infty(X, \mu)$ corresponding to the measure μ.*

(i) $w(0) = 1$, $w(\xi + \eta) = w(\xi)w(\eta)$ *for all* $\xi, \eta \in K$;

(ii) $\tau(w(\xi)) = \exp(-\|\xi\|^2)$ *for all* $\xi \in K$.

(iii) The linear span of $\{w(\xi), \xi \in K\}$ is a weakly dense subalgebra of the von Neumann algebra $L^\infty(X, \mu) \subset B(L^2(X, \mu))$.

Proof. (i) is trivial and (ii) follows from the calculation of the Fourier transform of a Gaussian random variable:

$$\tau(w(\xi)) = \frac{1}{\sqrt{2\pi}\|\xi\|}\int_{\mathbb{R}} e^{ix}e^{-x^2/(2\|\xi\|^2)}dx = \exp(-\|\xi\|^2).$$

To check property (iii), first note that since K is generating, the σ-subalgebra of $\mathcal{B}$ generated by the functions $w(\xi)$, $\xi \in K$ is $\mathcal{B}$ itself. This precisely means that the $w(\xi)$'s generate $L^\infty(X, \mathcal{B}, \mu)$ as a von Neumann algebra. □

Proposition II.1.4. *Assume that K is a real Hilbert space and that (A, τ) is a tracial von Neumann algebra generated by unitary elements $(w(\xi))_{\xi\in K}$ which satisfy $(i) - (iii)$ from Proposition II.1.3.*

Then the vectors $(w(\xi))_{\xi\in K}$ are linearly independent (over $\mathbb{C}$). Therefore, for any orthogonal operator $T \in \mathcal{O}(K)$, the equation $\sigma_T(w(\xi)) := w(T\xi)$, for all $\xi \in K$ uniquely defines an automorphism of (A, τ).

Proof. Assume that $\lambda_1, \cdots, \lambda_n \in \mathbb{C}$ and $\xi_n, \cdots, \xi_n \in K$ are such that

$$\sum_{i=1}^n \lambda_i w(\xi_i) = 0.$$

Then for any $\xi \in K$, multiplying this equality by $w(\xi)$ and taking the trace gives

$$0 = \sum_{i=1}^n \lambda_i \exp(-\|\xi_i + \xi\|^2) = \exp(-\|\xi\|^2)\sum_{i=1}^n \lambda_i' \exp(-2\langle \xi, \xi_i\rangle), \tag{II.1}$$

with $\lambda_i' = \lambda_i \exp(-\|\xi_i\|^2)$. Take a vector $\eta \in K$ such that $\langle \eta, \xi_i\rangle \neq \langle \eta, \xi_j\rangle$ for all $i \neq j$. Then the functions $(f_i)_{i=1}^n$ given by $f_i : t \in \mathbb{R} \mapsto \exp(-2t\langle \eta, \xi_i\rangle)$ are linearly independant. Using equation (II.1) for vectors ξ of the form $\xi = t\eta$, $t \in \mathbb{R}$, we see that $\lambda_i' = 0$ (hence $\lambda_i = 0$) for all $i \in \{1, \cdots, n\}$. □

Combining the above propositions, we can identify H to a generating Gaussian Hilbert space in some $L^2_{\mathbb{R}}(X_\pi, \mu)$ so that the representation π induces a trace preserving action $\Gamma \curvearrowright L^\infty(X_\pi, \mu)$. This is the Gaussian action σ_π, characterized by the following abstract description.

Definition II.1.5 (Universal description, [Va13])**.** Consider an orthogonal representation $\pi : \Gamma \to H_{\mathbb{R}}$. The *Gaussian action* σ_π associated with π is the unique action (up to conjugacy) $\Gamma \curvearrowright (A_\pi, \tau)$ such that

(i) (A_π, τ) is a tracial von Neumann algebra generated by unitaries $w(\xi)_{\xi\in H}$ satisfying $w(0) = 1$, $w(\xi + \eta) = w(\xi)w(\eta)$ and $\tau(w(\xi)) = \exp(-\|\xi\|^2)$ for all $\xi, \eta \in H$.

(ii) For all $g \in \Gamma$ and all $\xi \in H$, one has $\sigma_\pi(g)(w(\xi)) = w(\pi(g)\xi)$.

The uniqueness above is due to Proposition II.1.4, which implies as well that $\pi \mapsto \sigma_\pi$ is a functor from the category of orthogonal representations to the category of pmp actions. Note that this uniqueness implies that the Gaussian action does not depend on the embedding of H as a generating Gaussian Hilbert space.

Example II.1.6.
- If π is finite dimensional, the Gaussian action σ_π is as described at the beginning of this section.
- If $\pi : \Gamma \to \mathcal{O}(\ell^2(I))$ is the quasi-regular representation associated with an action $\Gamma \curvearrowright I$, then one easily checks that σ_π is the Bernoulli action $\Gamma \curvearrowright (\mathbb{R}^I, \mu)$ given by $g \cdot (x_i)_i = (x_{g^{-1} \cdot i})_i$. This observation will be further developed in Section II.1.11.

For later use, let us end this paragraph with a well known property of Gaussian Hilbert spaces.

Proposition II.1.7. *If $K \subset L^2_{\mathbb{R}}(X, \mu)$ is a Gaussian Hilbert space, then any orthonormal family $(f_i)_{i \in I}$ in K forms a family of independent standard Gaussian random variables.*

Proof. Since the square of the norm $\| \cdot \|_2$ of a random variable is equal to its variance, the f_i's are standard Gaussian random variables. We need to show that they are independent. Without loss of generality, we can assume that I is finite, say $I = \{1, \cdots, n\}$.

It is enough to check that the characteristic function ϕ_X of the random variable $X = (f_1, \cdots, f_n)$ is the product of the characteristic functions ϕ_i of the f_i's:

$$\phi_X(x_1, \cdots, x_n) = \phi_1(x_1) \cdots \phi_n(x_n), \ \forall (x_1, \cdots x_n) \in \mathbb{R}^n.$$

But we have

$$\phi_X(x_1, \cdots, x_n) = \int_X e^{i \sum_{k=1}^n x_k f_k} d\mu = \exp(-\frac{1}{2} \| \sum_{k=1}^n x_k f_k \|_2^2), \ \forall (x_1, \cdots, x_n) \in \mathbb{R}^n,$$

because $\sum_{k=1}^n x_k f_k$ belongs to K and hence is a centered Gaussian variable on (X, μ). So the result is clear since the family $(f_i)_{i \in I}$ is orthonormal. □

II.1.1.2 Operator algebraic description

Denote by $H_{\mathbb{C}} = H \otimes \mathbb{C}$ the complexified Hilbert space of H, and by $H_{\mathbb{C}}^{\odot n}$ the n'th symmetric tensor power. It is the closed subset of $H_{\mathbb{C}}^{\otimes n}$ spanned elements of the form

$$\xi_1 \odot \cdots \odot \xi_n := \frac{1}{n!} \sum_{\sigma \in S_n} \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(n)}, \ \xi_1, \cdots \xi_n \in H_{\mathbb{C}}.$$

We normalize inner product on $H_{\mathbb{C}}^{\odot n}$ in such a way that $\|\xi\|^2_{H_{\mathbb{C}}^{\odot n}} = n! \|\xi\|^2_{H_{\mathbb{C}}^{\otimes n}}$, for all $\xi \in H_{\mathbb{C}}^{\odot n}$.

Consider the *symmetric Fock space* associated with $H_{\mathbb{C}}$

$$\mathcal{S}(H_{\mathbb{C}}) := \mathbb{C}\Omega \oplus \bigoplus_{n \geq 1} H_{\mathbb{C}}^{\odot n}$$

The vector Ω is called the *vacuum vector* (or vacuum state).

Now, any vector $\xi \in H$ gives rise to an unbounded operator $x_\xi \in B(\mathcal{S}(H_{\mathbb{C}}))$, called a (symmetric) *creation operator*, defined on $\mathcal{S}(H_{\mathbb{C}})$ by the formulae

$$x_\xi(\Omega) = \xi, \ \text{and} \ x_\xi(\eta_1 \odot \cdots \odot \eta_n) = \xi \odot \eta_1 \odot \cdots \odot \eta_n.$$

Taking real part (times 2), we get an unbounded, self-adjoint operator

$$s(\xi) = x_\xi + x_\xi^*.$$

One checks that since H is a *real* Hilbert space, the operators $s(\xi)_{\xi \in H}$ commute to each other. Also, for $\xi \perp \eta$ and $n, m \in \mathbb{Z}$, we see that $\langle s(\xi)^n s(\eta)^m \Omega, \Omega \rangle = \langle s(\xi)^n \Omega, \Omega \rangle \langle s(\eta)^m \Omega, \Omega \rangle$. For these reasons, $s(\xi)$ and $s(\eta)$ may be regarded as independant random variables. In [PS12], it is moreover shown that the distribution of $s(\xi)$ with respect to the vacuum state is a Gaussian distribution $\mathcal{N}(0, \|\xi\|^2)$.

Consider the von Neumann subalgebra A of $B(\mathcal{S}(H_\mathbb{C}))$ generated by operators of the form $u(\xi_1, \cdots, \xi_k) = \exp(i\pi s(\xi_1) \cdots s(\xi_k))$, for $\xi_1, \cdots, \xi_k \in H$. Denote by τ the vector state on A associated with Ω. Then Peterson and Sinclair show the following.

Theorem II.1.8 ([PS12])**.** *The representation of A on $\mathcal{S}(H_\mathbb{C})$ is isomorphic to the GNS representation associated with τ. Moreover A is maximal abelian inside $B(\mathcal{S}(H_\mathbb{C}))$.*

Now note that any orthogonal operator $T \in \mathcal{O}(H)$ can be viewed as a unitary operator on $H_\mathbb{C}$, and thus gives rise to a unitary operator $\tilde{T} \in \mathcal{U}(\mathcal{S}(H_\mathbb{C}))$ such that

$$\tilde{T}(\Omega) = \Omega \text{ and } \tilde{T}(\xi_1 \odot \cdots \odot \xi_k) = (T\xi_1) \odot \cdots \odot (T\xi_k).$$

Then $\tilde{T} u(\xi_1, \cdots, \xi_k) \tilde{T}^* = u(T\xi_1, \cdots, T\xi_k)$, hence $\tilde{T}$ normalizes A. Since $\tilde{T}(\Omega) = \Omega$, $\mathrm{Ad}(\tilde{T})$ is a trace preserving automorphism of A.

Proposition II.1.9. *With the above notations, the Gaussian action σ_π associated with the representation π is conjugate with the action $\Gamma \curvearrowright^\sigma (A, \tau)$ defined by $\sigma_g = \mathrm{Ad}(\widetilde{\pi(g)})$ for all $g \in \Gamma$.*

Proof. This is obvious, because A is generated by the unitary elements $w(\xi) := u(\xi)$, $\xi \in H$, which satisfy the (i) of Definition II.1.5 and the action is such that $\sigma_g(w(\xi)) = w(\pi(g)\xi)$ for all $\xi \in H$. □

With this description of Gaussian actions, Theorem II.1.8 implies the following useful corollary.

Corollary II.1.10. *The Koopman representation of Γ on $L^2(A_\pi, \tau)$ is isomorphic to the representation $\tilde{\pi}$ on $\mathcal{S}(H)$ given by $\tilde{\pi}(g) = \widetilde{\pi(g)}$:*

$$L^2(A_\pi, \tau) \simeq \mathcal{S}(H_\mathbb{C}) \textit{ as } \Gamma\textit{-representations.}$$

II.1.1.3 Gaussian actions as twisted Bernoulli actions

This point of view allows to see directly that generalized Bernoulli shifts (with diffuse basis) are examples of Gaussian actions. The construction is *a priori* not very canonical, though: the correct initial data is rather an action of Γ on a countable set I, together with a Γ-invariant positive definite function on I (called the *covariance matrix*).

Consider a countable set I, and φ a positive definite function on I, that is, a symmetric map $\varphi : I \times I \to \mathbb{R}$ such that $\sum_{i,j \in F} x_i x_j \varphi(i,j) \geq 0$, for any finite subset $F \subset I$ and real numbers $x_i, i \in F$.

Then for any finite set $F \subset I$, the map $\varphi_{|F \times F}$ can be viewed as a positive matrix of size $|F|$. Hence, one can consider the centered Gaussian measure μ_F on $\mathbb{R}^F$ with covariance matrix given by $\varphi_{|F \times F}$.

Using Kolmogorov Consistency Theorem, there exists a measure μ_φ on $\mathbb{R}^I$ such that for any finite set $F \subset I$, the canoninal projection $\pi_F : \mathbb{R}^I \to \mathbb{R}^F$ satisfies $(\pi_F)_*\mu_\varphi = \mu_F$.

Now any action $\Gamma \curvearrowright I$ such that $\varphi(g \cdot i, g \cdot j) = \varphi(i, j)$, for all $i, j \in I$ gives rise to an action on $\mathbb{R}^I$ obtained by shifting coordinates. By definition of μ_φ, this action is measure-preserving.

Starting with our representation π on the real Hilbert space H, consider a Γ-invariant countable subset $I \subset H$ such that $\mathrm{span}(I)$ is dense in H. Now define a positive definite function φ on I by the formula $\varphi(i, j) := \langle i, j\rangle$, $i, j \in I$. Since π is an orthogonal representation, we indeed have $\varphi(\pi(g) \cdot i, \pi(g) \cdot j) = \varphi(i, j)$, for all $i, j \in I$. We denote by σ_I the corresponding shift action $\Gamma \curvearrowright (\mathbb{R}^I, \mu_\varphi)$.

Proposition II.1.11. *The action σ_I described above is conjugate with the Gaussian action σ_π. In particular it does not depend, up to conjugacy, on the choice of an invariant set $I \subset H$.*

Proof. By definition of φ, the map

$$\begin{array}{rcl} \Psi : H & \to & L^2(\mathbb{R}^I, \mu_\varphi) \\ \sum_{j\in F} \lambda_j j & \mapsto & ((x_i)_{i\in I} \mapsto \sum_{j\in F} \lambda_j x_j), \end{array}$$

(with $F \subset I$ finite) is a well defined isometric embedding of H as a generating Gaussian Hilbert space. Hence the vectors $w(\xi) = \exp(i\sqrt{2}\Psi(\xi))$ satisfy condition (i) of Definition II.1.5 and the shift action σ_I clearly satisfies $(\sigma_I)_g(w(\xi)) = w(\pi(g)\xi)$ for all $\xi \in H$. □

With this point of view, the following becomes obvious.

Example II.1.12. If π is of the form $\Gamma \to \mathcal{O}(\ell^2(I))$ for some action on a countable set $\Gamma \curvearrowright I$, the Gaussian action σ_π is the Bernoulli action $\Gamma \curvearrowright (\mathbb{R}^I, \mu)$.

In contrast with the above example, let us mention that this description can be rather vague when the measure is μ_φ is degenerated. For instance, it is not so easy to compute the Gaussian action associated with an irrational rotation of $\mathbb{R}^2$.

II.1.2 First properties

We show here how properties of a Gaussian action σ_π are related to properties of the initial representation π.

Lemma II.1.13. *Let π be a representation of Γ. Then $A_{\pi\oplus\pi} \simeq A_\pi \overline{\otimes} A_\pi$ and under this identification, $\sigma_{\pi\oplus\pi} = \sigma_\pi \otimes \sigma_\pi$.*

Proof. Note that $A_\pi \overline{\otimes} A_\pi$ is generated by the unitary elements $w(\xi) \otimes w(\eta)$, for $\xi, \eta \in H$, which satisfy the same relations as the $w(\xi \oplus \eta)$'s. Therefore the map $w(\xi \oplus \eta) \mapsto w(\xi) \otimes w(\eta)$, $\xi, \eta \in H$ extends to a *-isomorphism from $A_{\pi\oplus\pi}$ onto $A_\pi \overline{\otimes} A_\pi$, which intertwines the actions $\sigma_{\pi\oplus\pi}$ and $\sigma_\pi \otimes \sigma_\pi$. □

Note that the above lemma is clear for finite dimensional representations with the description given at the beginning of Section II.1.1.1.

Proposition II.1.14. *The Gaussian action σ_π associated with a representation π is essentially free if and only if π is faithful.*

Proof. If $\pi(g) = \mathrm{id}$ for some $g \in \Gamma$, then σ_g acts trivially on $\{w(\xi), \xi \in H\}$, hence on A_π. Conversely, assume that σ_g acts trivially on pA_π for some nonzero projection $p \in A_\pi$. Then for all $\xi \in H$, we have $pw(\pi(g)\xi) = pw(\xi)$, or equivalently $pw(\pi(g)\xi - \xi) = p$.

If $\pi(g)\xi \neq \xi$ for some $\xi \in H$ then the sequence $w(k(\pi(g)\xi - \xi))$, $k \geq 0$ converges weakly to zero. This contradicts the equality $pw(\pi(g)\xi - \xi) = p$. So $\pi(g)$ has to be the identity operator. ☐

Since we only work with free actions, **we will only consider faithful representations**.

In the following result, σ_π^0 denotes the unitary representation of Γ on $L^2(A_\pi, \tau) \ominus \mathbb{C}$ induced by σ_π. The proof follows from Corollary II.1.10.

Proposition II.1.15 ([PS12], Proposition 1.7)**.** *Let π a representation of Γ. Let $\mathcal{P}$ be any property in the following list:*

- *being mixing;*
- *being weakly contained in the left regular representation;*
- *having a tensor power which is weakly contained in the left regular representation.*

Then π has property $\mathcal{P}$ if and only if σ_π^0 does.

Since having no invariant vectors is not stable under tensor product, we need to replace this notion by its stable version in order to have a criterion of ergodicity.

Proposition II.1.16 ([PS12], Theorem 1.8)**.** *The Gaussian action σ_π is ergodic if and only if π is weakly mixing (meaning that $\pi \otimes \pi$ has no invariant vectors).*

This proposition admits an "approximate" version, regarding strong ergodicity. This is the purpose of the next section.

Using Proposition II.1.15, we see that many Gaussian actions are not conjugate to Bernoulli shift actions.

Corollary II.1.17. *If π is a mixing representation which is not weakly contained in the regular representation, the associated Gaussian action is not conjugate to a generalized Bernoulli action.*

Proof. If π is mixing and not weakly contained in the regular representation, then this is also the case of σ_π^0, by Proposition II.1.15. But a generalized Bernoulli action $\sigma : \Gamma \curvearrowright X^I$ is mixing if and only if the stabilizers of the action $\Gamma \curvearrowright I$ are finite. The latter implies that its koopman Representation σ^0 is weakly contained in the regular representation. Hence σ_π and σ are not conjugate. ☐

In order to have results on the von Neumann algebra associated with a Gaussian action, we will need however to make spectral hypothesis on the initial representation π. For instance we will assume that some tensor power of π is weakly contained in the regular representation. Strongly ℓ^p, $p > 2$ representations satisfy this hypothesis, and they are moreover mixing.

Definition II.1.18. Let $p \geq 2$. A representation $\pi : \Gamma \to \mathcal{O}(H)$ is said to be *strongly* ℓ^p if for any $\varepsilon > 0$, there exists a dense subspace $H_0 \subset H$ such that for all $\xi, \eta \in H_0$, the coefficient function $(\langle \pi(g)\xi, \eta\rangle)_g$ is in $\ell^{p+\varepsilon}(\Gamma)$.

As the following result shows, the class of Gaussian actions associated with strongly ℓ^p representations is larger than the class of generalized Bernoulli actions, for many interesting groups. We are gratefull to B. Bekka for pointing this out to us.

Proposition II.1.19 (Bekka)**.** *Every lattice Γ in a non-compact, simple Lie group G with finite center admits a unitary representation which is strongly ℓ^p for some $p > 2$, but not weakly contained in the regular representation.*

Proof. It is a known fact that G admits an irreducible representation π with no invariant vectors which is not strongly ℓ^q, for some $q > 2$. By [CHH88], π is not weakly contained in the regular representation of G. But by [Cow79, Théorème 2.4.2, Théorème 2.5.2], there exist a $p > 2$ such that π is strongly ℓ^p.
We check that $\pi_{|\Gamma}$ satisfies the proposition. It is easy to check that being strongly ℓ^p is stable by restriction to a lattice, so we are left to prove that $\pi_{|\Gamma}$ is not weakly contained in the left regular representation λ_Γ of Γ. Denote by λ_G the left regular representation of G.
Assume by contradiction that $\pi_{|\Gamma}$ is weakly contained in λ_Γ. Then by stability of weak containment under induction, we get that $\mathrm{Ind}_\Gamma^G(\pi_{|\Gamma})$ is weakly contained in $\lambda_G = \mathrm{Ind}_\Gamma^G(\lambda_\Gamma)$. However, $\mathrm{Ind}_\Gamma^G(\pi_{|\Gamma}) = \pi \otimes \mathrm{Ind}_\Gamma^G(1_\Gamma)$, and since Γ has finite co-volume in G, the trivial G-representation is contained in $\mathrm{Ind}_\Gamma^G(1_\Gamma) = \lambda_{G/\Gamma}$. Altogether, we get that π is weakly contained in λ_G, which is absurd. □

II.1.3 Strong ergodicity for Gaussian actions

Definition II.1.20. A measure-preserving action $\Gamma \curvearrowright (X, \mu)$ on a probability space is said to be *strongly ergodic* if every sequence of almost invariant subsets $(A_n)_{n\in\mathbb{N}}$ of X is trivial:

$$\left(\lim_n \mu((g \cdot A_n)\Delta A_n) = 0 \text{ for all } g \in \Gamma\right) \Rightarrow \lim_n \mu(A_n)(1 - \mu(A_n)) = 0.$$

Here is a very standard characterization of strong ergodicity.

Proposition II.1.21. *A pmp action $\Gamma \curvearrowright (X, \mu)$ is strongly ergodic if and only if every asymptotically invariant bounded sequence $(x_n)_n \subset L^\infty(X)$ (*i.e. $\lim_\omega \|\sigma_g(x_n) - x_n\|_2 = 0$ *for all $g \in \Gamma$) is trivial:* $\lim_n \|x_n - \tau(x_n)\|_2 = 0$*, where τ is the trace on $L^\infty(X)$ corresponding to μ.*

Proof. Fix a free ultrafilter ω on $\mathbb{N}$. Put $M := L^\infty(X, \mu) \rtimes \Gamma$. By definition, $\Gamma \curvearrowright (X, \mu)$ is strongly ergodic if and only if $M' \cap L^\infty(X, \mu)^\omega$ admits no non-trivial projection. This is of course equivalent to saying that this von Neumann algebra is trivial, which exactly means that asymptotically invariant bounded sequences in $L^\infty(X, \mu)$ are trivial. □

The following criterion for strong ergodicity of Gaussian actions generalizes the main result of [KT08]. It shows that for Gaussian actions, the so-called spectral gap property (property (ii) below) is equivalent to strong ergodicity. The proof is essentially the same as the one of [PS12, Theorem 1.8], combined with a standard Powers-Størmer argument, and some calculations about Gaussian random variables.

Note that the equivalence between (i) and (ii) was already pointed out in [Po08].

Theorem II.1.22. *Let $\pi : \Gamma \to \mathcal{O}(H)$ be an orthogonal representation, and denote by σ_π the associated Gaussian action and by σ_π^0 the Koopman representation on $L^2(X_\pi) \ominus \mathbb{C}1$. Then the following are equivalent.*

(i) π is not amenable, in the sense that $\pi \otimes \pi$ does not admit almost invariant vectors ([Be90]);

(ii) σ_π^0 does not admit almost invariant vectors;

(iii) σ_π is strongly ergodic.

We start with three lemmas.

Lemma II.1.23. *Assume that H is a real Hilbert space and that $s : H \to L^2_{\mathbb{R}}(X, \mu)$ is an embedding as a Gaussian Hilbert space. Consider the real Hilbert space $H \odot H$ with renormalized inner product such that $\|\xi\|^2_{H\odot H} = 2\|\xi\|^2_{H\otimes H}$, for all $\xi \in H \odot H$. Then the map*

$$\begin{array}{rccc} s^{(2)}: & H \odot_{alg} H & \to & L^2_{\mathbb{R}}(X,\mu) \\ & \xi \odot \eta & \mapsto & s(\xi)s(\eta) - \langle \xi, \eta\rangle 1 \end{array}$$

extends to an isometry $s^{(2)} : H \odot H \to L^2_{\mathbb{R}}(X, \mu)$.

Proof. The map is clearly well defined on $H \odot_{\text{alg}} H = \text{span}\{\xi \odot \eta\,,\, \xi, \eta \in H\}$. We need to show that it intertwines inner products. Fix $(\delta_i)_{i\in I}$ an orthonormal basis of H. It is sufficient to check that for all $i, j, i', j' \in I$,

$$\langle s(\delta_i)s(\delta_j) - \langle \delta_i, \delta_j\rangle 1, s(\delta_{i'})s(\delta_{j'}) - \langle \delta_{i'}, \delta_{j'}\rangle 1\rangle_{L^2(X)} = \langle \delta_i \odot \delta_j, \delta_{i'} \odot \delta_{j'}\rangle.$$

But the right term above is equal to 2 if $i = j = i' = j'$, to 1 if $\{i, j\} = \{i', j'\}$ and $i \neq j$ and it is equal to 0 otherwise. The left hand side is easily seen to take the same values because the $s(\delta_i)$'s are independent standard Gaussian random variables, by Proposition II.1.7 (recall that $\int_{\mathbb{R}} t^2 d\nu(t) = 1$ and $\int_{\mathbb{R}} t^4 d\nu(t) = 3$, where ν is the standard Gaussian measure on $\mathbb{R}$). □

Lemma II.1.24. *Let (X, μ) be a standard probability space, and let $f, g \in L^2_{\mathbb{R}}(X, \mu)$ be real valued functions. Then we have $\|e^{if} - e^{ig}\|_2 \leq \|f - g\|_2$.*

Proof. From the fact that $2 - 2\cos(u) \leq u^2$ for all $u \in \mathbb{R}$, we see that

$$\|e^{if} - e^{ig}\|_2^2 = 2 - 2\Re\Big(\int_X e^{i(f-g)} d\mu\Big) = \int_X (2 - 2\cos(f - g))d\mu \leq \|f - g\|_2^2. \qquad \square$$

The following lemma provides a control of the norm $\|\cdot\|_4$ on a specific subspace of $L^2_{\mathbb{R}}(X_\pi)$ in terms of the norm $\|\cdot\|_2$. This Lemma was proved in [Ja97] (Lemma 3.44 therein) using the so-called Wick products. For completeness, we will provide an elementary proof after the proof of Theorem II.1.22.

Lemma II.1.25. *Assume that $H_{\mathbb{R}}$ is a real Hilbert space and that $s : H \to L^2_{\mathbb{R}}(X, \mu)$ is an embedding as a Gaussian Hilbert space. With the notations of Lemma II.1.23, put $K^{(2)} = s^{(2)}(H \odot H)$.*

There exists a constant $C > 0$ such that $\|h\|_4 \leq C\|h\|_2$, for all $h \in K^{(2)}$.

Proof of Theorem II.1.22. $(i) \Rightarrow (ii)$. As Γ-representations, we have by Corollary II.1.10 that

$$\sigma_\pi^0 \simeq \bigoplus_{n\geq 1} \pi_{\mathbb{C}}^{\odot n} \subset \bigoplus_{n\geq 1} \pi_{\mathbb{C}}^{\otimes n},$$

where $\pi_{\mathbb{C}}$ denotes the complexificaction of π. But $\bigoplus_{n\geq 1} \pi_{\mathbb{C}}^{\otimes n}$ is of the form $\pi_{\mathbb{C}} \otimes \rho$, for some representation ρ. Hence if σ_π^0 has almost invariant vectors, this is also the case of $\pi \otimes \pi$ (see for instance [Po08, Lemma 3.2]).

$(ii) \Rightarrow (iii)$ is trivial, and true for any pmp action.

$(iii) \Rightarrow (i)$. Assume that $\pi \otimes \pi$ has almost invariant vectors, *i.e.* unit vectors $(\xi_n)_n \in H \otimes H$ such that $\lim_n \|(\pi(g) \otimes \pi(g))\xi_n - \xi_n\| = 0$ for all $g \in \Gamma$. View the vectors ξ_n as Hilbert Schmidt operators on H, and for all $n \in \mathbb{N}$ put $\eta_n = (\xi_n^* \xi_n)^{1/2}$. As pointed out in the proof of [PS12, Theorem 1.8], these vectors belong to $H \odot H$. They are still unit vectors.

Furthermore, the (real) space of Hilbert-Schmidt operators $\mathrm{HS}(H)$ embeds isometrically in $\mathrm{HS}(H_{\mathbb{C}})$. Hence, using the Powers-Størmer inequality inside $\mathrm{HS}(H_{\mathbb{C}})$, we see that for any $g \in \Gamma$ and $n \in \mathbb{N}$,

$$\|(\pi(g) \otimes \pi(g))\eta_n - \eta_n\|_{\mathrm{HS}}^2 \leq 2\|(\pi(g) \otimes \pi(g))\xi_n - \xi_n\|_{\mathrm{HS}}.$$

So the η_n's are almost invariant, unit vectors in $H \odot H$.

Now with the notations of Lemma II.1.23, define a sequence of unitaries $u_n \in L^\infty(X_\pi, \mu)$ by $u_n = \exp(is^{(2)}(\alpha\eta_n))$ for all n, where $\alpha > 0$ will be chosen later independently of n. Let us check that $(u_n)_n$ is a non-trivial, asymptotically invariant sequence in $L^\infty(X_\pi)$.

By Lemma II.1.24, the sequence $(u_n)_n$ is asymptoticaly Γ-invariant because the vectors η_n are almost invariant.

Now fix $\eta \in H \odot H$ such that $\|\eta\| \leq 1$. Note that for all $u \in \mathbb{R}$, we have $1 - u^2/2 \leq \cos(u) \leq 1 - u^2/2 + u^4/24$. Hence

$$1 - \frac{1}{2}\int_{X_\pi} s^{(2)}(\eta)^2 d\mu \leq \Re(\tau(\exp(is^{(2)}(\eta))) \leq 1 - \frac{1}{2}\int_{X_\pi} s^{(2)}(\eta)^2 d\mu + \frac{1}{24}\int_{X_\pi} s^{(2)}(\eta)^4 d\mu.$$

Together with Lemmas II.1.23 and II.1.25, this implies

$$1 - \frac{1}{2}\|\eta\|^2 \leq \Re(\tau(\exp(is^{(2)}(\eta))) \leq 1 - \frac{1}{2}\|\eta\|^2 + \frac{C^4}{24}\|\eta\|^4.$$

So we obtain

$$|\Re(\tau(\exp(is^{(2)}(\eta)))| \leq 1 - \frac{1}{2}\|\eta\|^2 + \frac{C^4}{24}\|\eta\|^4. \tag{II.2}$$

Similarly we can bound the imaginary part by using the fact that $u - u^3/6 \leq \sin(u) \leq u$ for all $u \in \mathbb{R}$. Since the variables in $s^{(2)}(H \odot H)$ are centered variables, we get

$$-\frac{1}{6}\int_{X_\pi} s^{(2)}(\eta)^3 d\mu \leq \Im(\tau(\exp(is^{(2)}(\eta))) \leq 0.$$

Using Cauchy-Schwarz inequality and Lemma II.1.25, this gives the bound

$$|\Im(\tau(\exp(is^{(2)}(\eta)))| \leq \frac{1}{6}\int_{X_\pi} |s^{(2)}(\eta)^3| d\mu \leq \frac{1}{6}\|s^{(2)}(\eta)\|_2 \|s^{(2)}(\eta)\|_4^2 \leq \frac{C^2}{6}\|\eta\|^3. \tag{II.3}$$

Now we apply II.2 and II.3 to the vectors $\alpha\eta_n$. We get for all n

$$\begin{aligned}|\tau(\exp(is^{(2)}(\alpha\eta_n))|^2 &= |\Re(\tau(\exp(is^{(2)}(\eta)))|^2 + |\Im(\tau(\exp(is^{(2)}(\eta)))|^2 \\ &\leq (1 - \frac{1}{2}\alpha^2 + \frac{C^4}{24}\alpha^4)^2 + \frac{C^4}{36}\alpha^6.\end{aligned}$$

So if we choose α small enough, the constant $c = (1 - \frac{1}{2}\alpha^2 + \frac{C^4}{24}\alpha^4)^2 + \frac{C^4}{36}\alpha^6$ is less than one. In that case, we get

$$\|u_n - \tau(u_n)\|_2^2 = 1 - |\tau(\exp(is^{(2)}(\alpha\eta_n))|^2 \geq 1 - c > 0.$$

This shows that the sequence (u_n) is non-trivial and the action is not strongly ergodic by Proposition II.1.21. □

It remains to prove Lemma II.1.25.

Proof of Lemma II.1.25. From the definition of $s^{(2)}$, we see $K^{(2)}$ is the closed linear span of the set $\{fg - \int_X fg d\mu\,,\, f, g \in K\}$, where $K = s(H)$.

Take an orthonormal basis $(f_i)_{i\in\mathbb{N}}$ of K: the f_i's are independent standard Gaussian random variables, by Proposition II.1.7. Decompose $K^{(2)}$ as an orthogonal direct sum $K^{(2)} = \overline{K_1} \oplus \overline{K_2}$, with

- $K_1 = \operatorname{span}\{f_i^2 - 1\,,\, i \in \mathbb{N}\}$,
- $K_2 = \operatorname{span}\{f_i f_j\,,\, i, j \in \mathbb{N}, i \neq j\}$.

Step 1. There exists a constant $C_1 > 0$ such that $\|h\|_4 \leq C_1\|h\|_2$ for all $h \in K_1$.

Take $h \in K_1$ and write $h = \sum_{i\in F}\lambda_i(f_i^2 - 1)$, with $F \subset \mathbb{N}$ finite and $\lambda_i \in \mathbb{R}$. Note that

$$\|h\|_2^2 = \sum_{i\in F}\lambda_i^2\int_X (f_i^2 - 1)^2 d\mu.$$

On the other hand

$$\int_X h^4 d\mu = \sum_{i,j,k,l\in F}\lambda_i\lambda_j\lambda_k\lambda_l\int_X (f_i^2 - 1)(f_j^2 - 1)(f_k^2 - 1)(f_l^2 - 1)d\mu.$$

But such an integral $\int_X(f_i^2 - 1)(f_j^2 - 1)(f_k^2 - 1)(f_l^2 - 1)d\mu$ is equal to 0 whenever one of the indices i, j, k, l is different from the others. So the only nonzero terms in the above sum are the ones for which either $i = j$ and $k = l$, or $i = k$ and $j = l$, or $i = l$ and $j = k$. Hence we get:

$$\begin{aligned}\|h\|_4^4 \leq &\sum_{i=j,k=l\in F}\lambda_i^2\lambda_k^2\int_X (f_i^2 - 1)^2(f_k^2 - 1)^2 d\mu + \sum_{i=k,j=l\in F}\lambda_i^2\lambda_j^2\int_X (f_i^2 - 1)^2(f_j^2 - 1)^2 d\mu \\ &+ \sum_{i=l,j=k\in F}\lambda_i^2\lambda_j^2\int_X (f_i^2 - 1)^2(f_j^2 - 1)^2 d\mu \\ \leq &\, 3M\sum_{i,j\in F}\lambda_i^2\lambda_j^2\int_X (f_i^2 - 1)^2 d\mu\int_X (f_j^2 - 1)^2 d\mu = 3M\|h\|_2^4,\end{aligned}$$

where $M = \max(1, \frac{\int_X (f_i^2-1)^4 d\mu}{(\int_X (f_i^2-1)^2 d\mu)^2})$ is independent of i because the (f_i)'s are identically distributed.

So we can put $C_1 = (3M)^{1/4}$, which does not depend on $h \in K_1$.

Step 2. There exists a constant $C_2 > 0$ such that $\|h\|_4 \leq C_2 \|h\|_2$ for all $h \in K_2$.

Consider $h \in K_2$, $h = \sum_{i,j \in F, i<j} \lambda_{i,j} f_i f_j$, with $F \subset \mathbb{N}$ finite, and $\lambda_{i,j} \in \mathbb{R}$ for all $i < j$. For convenience, define also $\lambda_{i,j} := 0$ for $i > j$. On the one hand, we have

$$\|h\|_2^2 = \int_X h^2 d\mu = \sum_{i \neq j} \lambda_{i,j}^2.$$

On the other hand we have

$$\int_X h^4 d\mu = \sum_{\substack{i_1 \neq j_1, i_2 \neq j_2 \\ i_3 \neq j_3, i_4 \neq j_4}} \prod_{s=1}^{4} \lambda_{i_s,j_s} \int_X \prod_{s=1}^{4} f_{i_s} f_{j_s} d\mu.$$

But if for some $k \in F$, there is an odd number of coordinates of $\bar{i} := (i_1, j_1, i_2, j_2, i_3, j_3, i_4, j_4)$ which are equal to k, then the integral $\int_X \prod_{s=1}^4 f_{i_s} f_{j_s} d\mu$ is equal to zero, because the odd moments of f_k are equal to zero. So if $\int_X \prod_{s=1}^4 f_{i_s} f_{j_s} d\mu$ is non-zero then the coordinates of $\bar{i}$ can be grouped by pairs.

This simple observation first allows to say that $\int_X \prod_{s=1}^4 f_{i_s} f_{j_s} d\mu$ takes values in the set

$$\{\int_{\mathbb{R}} t^8 d\nu(t), (\int_{\mathbb{R}} t^4 d\nu(t))^2, (\int_{\mathbb{R}} t^2 d\nu(t))^4, (\int_{\mathbb{R}} t^2 d\nu(t))^2 \int_{\mathbb{R}} t^4 d\nu(t), 0\} = \{105, 9, 1, 3, 0\}$$

(ν is the standard Gaussian measure on $\mathbb{R}$). So $\int_X \prod_{s=1}^4 f_{i_s} f_{j_s} d\mu$ is at most equal to 105.

It also allows to describe the 8-uplets $\bar{i}$ for which the integral $\int_X \prod_{s=1}^4 f_{i_s} f_{j_s} d\mu$ is non-zero. It is easily checked that up to permutation of the four pairs (i_s, j_s) and up to permutation of i_s with j_s for some s's, the integral is zero unless $\bar{i}$ belongs to one of the sets

- $I_1 = \{(a, b, a, b, c, d, c, d),\ a, b, c, d \in F,\ a \neq b,\ c \neq d\}$
- $I_2 = \{(a, b, b, c, c, d, d, a),\ a, b, c, d \in F,\ a \neq b,\ b \neq c,\ c \neq d,\ d \neq a\}$.

Denote by $\mathcal{S} \subset S_8$ the subgroup of permutations of $\{1, \cdots, 8\}$ which is generated by the transposition $(1, 2)$ and by the permutations of the four pairs $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$ and $\{7, 8\}$. For any $\sigma \in \mathcal{S}$, denote by $\sigma(I_1) = \{\sigma(\bar{i}),\ \bar{i} \in I_1\}$, and define similarly $\sigma(I_2)$.

Altogether, we get

$$\int_X h^4 \leq 105 \sum_{\sigma \in \mathcal{S}} \left(\sum_{\bar{i} \in \sigma(I_1)} \prod_{s=1}^{4} |\lambda_{i_s,j_s}| + \sum_{\bar{i} \in \sigma(I_2)} \prod_{s=1}^{4} |\lambda_{i_s,j_s}| \right). \tag{II.4}$$

Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \sum_{\bar{i} \in I_2} |\lambda_{i_1,j_1} \lambda_{i_2,j_2} \lambda_{i_3,j_3} \lambda_{i_4,j_4}| &= \sum_{a,b,c,d \in F} |\lambda_{a,b} \lambda_{b,c} \lambda_{c,d} \lambda_{d,a}| \\ &\leq (\sum_{a,b,c,d} \lambda_{a,b}^2 \lambda_{c,d}^2)^{1/2} (\sum_{a,b,c,d} \lambda_{b,c}^2 \lambda_{d,a}^2)^{1/2} = \|h\|_2^4. \end{aligned}$$

And actually, exactly the same calculation shows that for any $\sigma \in \mathcal{S}$,

$$\sum_{\bar{i}\in\sigma(I_2)} \prod_{s=1}^{4} |\lambda_{i_s,j_s}| \leq \|h\|_2^4.$$

Similarly, for any $\sigma \in \mathcal{S}$, the sum $\sum_{\bar{i}\in\sigma(I_1)} \prod_{s=1}^{4} |\lambda_{i_s,j_s}|$ is equal to the product of two sums of the form $\sum_{a\neq b} \lambda_{a,b}^2$ or $\sum_{a\neq b} |\lambda_{a,b}\lambda_{b,a}| \leq \sum_{a\neq b} \lambda_{a,b}^2$. So we see that

$$\sum_{\bar{i}\in\sigma(I_1)} \prod_{s=1}^{4} |\lambda_{i_s,j_s}| \leq \|h\|_2^4.$$

Thus, Equation (II.4) implies that $\|h\|_4^4 \leq 105|\mathcal{S}|\|h\|_2^4 \leq 105(8!)\|h\|_2^4$. This proves Step 2.

By density, we see that for $i = 1, 2$ and $h \in \overline{K_i}$, we have $\|h\|_4 \leq C_i\|h\|_2$.

Step 3. Conclusion.

Since in $\mathbb{R}^2$ all norms are equivalent, there exists a constant $C > 0$ such that

$$C_1|a| + C_2|b| \leq C(a^2 + b^2)^{1/2}, \text{ for all } a, b \in \mathbb{R}.$$

Assume that $h = h_1 + h_2$, with $h_1 \in \overline{K_1}$ and $h_2 \in \overline{K_2}$. Then we have

$$\begin{aligned}\|h\|_4 &\leq \|h_1\|_4 + \|h_2\|_4 \\ &\leq C_1\|h_1\|_2 + C_2\|h_2\|_2 \\ &\leq C(\|h_1\|_2^2 + \|h_2\|_2^2)^{1/2} = C\|h\|_2.\end{aligned}$$

□

Remark II.1.26. It is much easier to show that the action is not strongly ergodic whenever π itself admits almost invariant unit vectors (ξ_n). Indeed, in that case set $u_n = \exp(i\sqrt{2}s(\xi_n))$ for all n. Then the trace of these unitaries is equal to $\exp(-\|\xi_n\|^2) = e^{-1}$, so the sequence is clearly non-trivial and asymptotically invariant.

Note also that for a quasi-regular representation π, being amenable and admitting almost-invariant unit vectors is equivalent. So our result is consistent with [KT08, Theorem 1.2].

II.2 Deformation/rigidity results

In this section we prove results regarding the position of "rigid" subalgebras of the crossed-product von Neumann algebra $M = A \rtimes \Gamma$ associated with a Gaussian action $\sigma : \Gamma \curvearrowright A$. This is performed by applying Popa's deformation/rigidity theory, as follows.

1. In a first step we explain, following [PS12], how to construct an interesting one-parameter group $(\alpha_t)_t$ of automorphisms of a von Neumann algebra which contains M.

2. Then we give sufficient conditions for a subalgebra $Q \subset M$ to be (α_t)-*rigid*, in the sense that $\lim_{t\to 0} \|\alpha_t(x) - x\|_2 = 0$, uniformly in $x \in \mathcal{U}(Q)$.

3. Finally we will use techniques due to Popa ([Po06a, Po06b, Po06c]) to describe the position of rigid subalgebras of M. Their position will be compared to the crossed product decomposition $M = A \rtimes \Gamma$.

II.2.1 Deformation of Gaussian actions

We are interested in malleability properties of Gaussian actions, in the following sense.

Definition II.2.1 ([Po08]). A measure preserving action $\Gamma \curvearrowright (X, \mu)$ is said to be *s-malleable* if there exists a one-parameter group $(\alpha_t)_{t\in\mathbb{R}}$ of automorphisms of $L^\infty(X \times X, \mu \otimes \mu)$, and an automorphism $\beta \in \mathrm{Aut}(L^\infty(X \times X, \mu \otimes \mu))$ such that:

- the map $t \mapsto \alpha_t(x)$ is strongly continuous for any $x \in L^\infty(X \times X)$;
- the automorphisms α_t, $t \in \mathbb{R}$ and β commute with the double action of Γ on $X \times X$;
- $\alpha_1(L^\infty(X) \otimes 1) = 1 \otimes L^\infty(X)$, where we identify $L^\infty(X \times X) \simeq L^\infty(X) \overline{\otimes} L^\infty(X)$;
- for any $t \in \mathbb{R}$, one has $\alpha_t \circ \beta = \beta \circ \alpha_{-t}$;
- β acts trivially on $L^\infty(X) \otimes 1$ and $\beta^2 = \mathrm{id}$.

Such a pair $((\alpha_t)_t, \beta)$ is called an *s-malleable deformation* of the action.

As explained in [Fu07] or [PS12], Gausssian actions are s-malleable. The construction of a malleable deformation can be performed as follows.

Let $\pi : \Gamma \to \mathcal{O}(H)$ be an orthogonal representation of a group Γ on a real Hilbert space H. Denote by $\sigma : \Gamma \curvearrowright A$ the associated Gaussian action. By Lemma II.1.13, $\sigma \otimes \sigma$ is the Gaussian action associated with $\pi \oplus \pi$.
Define on $H \oplus H$ the block operators

$$\rho = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \theta_t = \begin{pmatrix} \cos(\pi t/2) & -\sin(\pi t/2) \\ \sin(\pi t/2) & \cos(\pi t/2) \end{pmatrix},\ t \in \mathbb{R}.$$

Here are some trivial facts about these operators:

- $\forall t \in \mathbb{R}$, $\rho \circ \theta_t = \theta_{-t} \circ \rho$;
- θ_t and ρ commute with $(\pi \oplus \pi)(g)$ for all $g \in \Gamma$, $t \in \mathbb{R}$;
- $\forall s, t \in \mathbb{R}$, $\theta_s \circ \theta_t = \theta_{t+s}$.

By Proposition II.1.4, ρ and (θ_t) induce respectively an automorphism β and a one-parameter group (α_t) of automorphisms of $A \overline{\otimes} A$ which are easily seen to be an s-malleable deformation of σ.

Now consider the crossed-product von Neumann algebras $M = A \rtimes \Gamma$ and $\tilde{M} = (A \overline{\otimes} A) \rtimes_{\sigma\otimes\sigma} \Gamma$. View M as a subalgebra of $\tilde{M}$ using the identification $M \simeq (A \overline{\otimes} 1) \rtimes \Gamma$. The automorphisms defined above then extend to automorphisms of $\tilde{M}$ still denoted (α_t) and β, in such a way that $\alpha_t(u_g) = \beta(u_g) = u_g$, for all $g \in \Gamma$.

Now that we have introduced a deformation of M, we can define the concept of rigid subalgebra (relatively to this deformation).

Definition II.2.2. For any projection $p \in M$, we will say that a von Neumann subalgebra $Q \subset pMp$ is (α_t)*-rigid* if the deformation $(\alpha_t)_t$ converges to the identity uniformly on the unit ball $(Q)_1$ of Q:

$$\lim_{t\to 0} \sup_{x\in(Q)_1} \|\alpha_t(x) - x\|_2 = 0.$$

The following lemma will be useful to check that some subalgebras are (α_t)-rigid. It is the so-called *transversality* Lemma.

Lemma II.2.3 ([Po08], Lemma 2.1)**.** *For any $x \in M$ and $t \in \mathbb{R}$ one has*

$$\|x - \alpha_{2t}(x)\|_2 \leq 2\|\alpha_t(x) - E_M \circ \alpha_t(x)\|_2.$$

Proof. Take $x \in M$ and $t \in \mathbb{R}$. Recall that $\beta \circ \alpha_{-t} = \alpha_t \circ \beta$ and that $\beta_{|M} = \mathrm{id}$. We have

$$\begin{aligned}
\|x - \alpha_{2t}(x)\|_2 &= \|\alpha_{-t}(x) - \alpha_t(x)\|_2 \\
&\leq \|\alpha_{-t}(x) - E_M \circ \alpha_t(x)\|_2 + \|E_M \circ \alpha_t(x) - \alpha_t(x)\|_2 \\
&= \|\beta \circ \alpha_{-t}(x) - \beta \circ E_M \circ \alpha_t(x)\|_2 + \|E_M \circ \alpha_t(x) - \alpha_t(x)\|_2 \\
&= 2\|\alpha_t(x) - E_M \circ \alpha_t(x)\|_2.
\end{aligned}$$

□

II.2.2 Examples of (α_t)-rigid subalgebras

General rigidity condition: property (T)

We consider a Gaussian action σ and we use the notations and definitions of Section II.2.1.

A sufficient condition for a subalgebra $Q \in M$ to be (α_t)-rigid is to have property (T), in the sense of [CJ85]. More generally, $Q \subset M$ is (α_t)-rigid if it satisfies a relative version of property (T), which can be formulated as follows in the setting of tracial von Neumann algebras.

Definition II.2.4 ([Po06c], Definition 4.2.1)**.** Let (N, τ) be a tracial von Neumann algebra, and B be a von Neumann subalgebra of N. The inclusion $B \subset N$ has *relative property (T)*, or is *rigid* if for any $\varepsilon > 0$, there exists $\delta > 0$ and $x_1, \cdots, x_n \in N$ with the following property.

Any normal, completely positive map $\phi : N \to N$ for which $\phi(1) \leq 1$, $\tau \circ \phi \leq \tau$ and $\|\phi(x_i) - x_i\|_2 \leq \delta$, $i = 1, \cdots, n$ automatically satisfies $\|\phi(x) - x\|_2 \leq \varepsilon$, for all $x \in (B)_1$.

Generalizing [CJ85, Theorem 2], Popa showed in [Po06c] that this notion coincides with the notion of relative property (T) for groups: for any countable groups $\Lambda < \Gamma$, the inclusion $L\Lambda \subset L\Gamma$ as property (T) if and only if the pair (Γ, Λ) has relative property (T).

With Definition II.2.4, we see easily that rigid inclusions $Q \subset M$ are (α_t)-rigid. Let us now give another condition, more specific to $\tilde{M}$ and (α_t), which ensures (α_t)-rigidity. It is the so-called *spectral gap rigidity*, discovered by Popa in [Po08].

Spectral gap rigidity

The following criterion is the main result of this section. It was proved by Popa [Po08] for Bernoulli actions. The idea to require that some tensor power of π is weakly contained in the regular representation and not necessarily π itself is due to Sinclair [Si11]. It allows to cover strongly ℓ^p-representations, as discussed at the end of Section II.1.2.

Proposition II.2.5 (Spectral gap rigidity)**.** *Assume that $\pi : \Gamma \to \mathcal{O}(H_{\mathbb{R}})$ is an orthogonal representation such that $\pi^{\otimes k}$ is weakly contained in the regular representation for some $k \geq 1$. Denote by $\sigma_\pi : \Gamma \curvearrowright (A, \tau)$ the associated Gaussian action and put $M = A \rtimes \Gamma$. Define $\tilde{M}$ and $(\alpha_t)_t \in \mathrm{Aut}(\tilde{M})$ as in Section II.2.1.*

If $Q \subset M$ is a von Neumann subalgebra with no amenable direct summand, then $Q' \cap M$ is (α_t)-rigid.

In fact we will rather use the follwing "corner version" of Proposition II.2.5.

Corollary II.2.6. *With the same hypotheses and notations as in Proposition II.2.5, if $p \in M$ is a projection and if $Q \subset pMp$ is a von Neumann subalgebra with no amenable direct summand, then $Q' \cap pMp$ is (α_t)-rigid.*

Proof. Since Q is non-amenable, then M is non amenable. So Γ is non-amenable, and M has no amenable direct summand. Thus the algebra $Q_1 := Q \oplus (1-p)M(1-p)$ has no amenable direct summand and we can apply Proposition II.2.5 to Q_1. The corollary follows because $Q' \cap pMp = p(Q_1' \cap M)p$. □

Proposition II.2.5 will be deduced from the following two lemmas.

Lemma II.2.7. *If $\pi^{\otimes k}$ is weakly contained in the regular representation for some $k \geq 1$, then the M-M-bimodule $\mathcal{H} = L^2(\tilde{M}) \ominus L^2(M)$ is such that the Connes' fusion tensor power of ${}_M\mathcal{H}_M$, $\mathcal{H}^{\otimes_M k} := \mathcal{H} \otimes_M \cdots \otimes_M \mathcal{H}$ is weakly contained in the coarse bimodule $L^2(M) \otimes L^2(M)$.*

Proof. As in the proof of [Va13, Lemma 3.5], for any representation $\eta : \Gamma \to \mathcal{U}(K)$, define an M-M bimodule structure $\mathcal{H}^\eta$ on the Hilbert space $K \otimes L^2(M)$ by

$$(au_g) \cdot (\xi \otimes x) \cdot (bu_h) = \eta_g(\xi) \otimes au_g x b u_h, \text{ for all } a, b \in A,\, g, h \in \Gamma,\, x \in M,\, \xi \in K.$$

We can make three observations regarding this definition.

- Weak containment for two representations η_1, η_2 of Γ implies the weak containment of the correpsonding bimodules $\mathcal{H}^{\eta_1}$, $\mathcal{H}^{\eta_2}$;
- If η is the regular representation then $\mathcal{H}^\eta$ is weakly contained in $L^2(M) \otimes L^2(M)$, because the A-A bimodule $L^2(A)$ is weakly contained in $L^2(A) \otimes L^2(A)$ (A is amenable);
- For two representation η_1, η_2, $\mathcal{H}^{\eta_1} \otimes_M \mathcal{H}^{\eta_2} = \mathcal{H}^{\eta_1 \otimes \eta_2}$.

Now remark that the M-M bimodule $L^2(\tilde{M}) \ominus L^2(M)$ is isomorphic to $\mathcal{H}^{\sigma_\pi^0}$. Moreover, Proposition II.1.15 implies that $(\sigma_\pi^0)^{\otimes k}$ is weakly contained in the regular representation. So the lemma follows from the above observations. □

Lemma II.2.8. *Let ω be a free ultrafilter on $\mathbb{N}$. If $\pi^{\otimes k}$ is weakly contained in the regular representation for some $k \geq 1$ then for every subalgebra $Q \subset M$ with no amenable direct summand, one has*

$$Q' \cap \tilde{M}^\omega \subset M^\omega.$$

Proof. By Lemma II.2.7, we know that $\mathcal{H}^{\otimes_M k}$ is weakly contained in the coarse M-M bimodule, were $\mathcal{H} = L^2(\tilde{M}) \ominus L^2(M)$. Now note that if $H^{\otimes_M K}$ is weakly contained in the coarse M-M bimodule, then this is also the case of $H^{\otimes_M K+1}$. Hence one can assume that k is of the form $k = 2^p$, which will be used later.

Fix $Q \subset M$ such that $Q' \cap \tilde{M}^\omega \not\subset M^\omega$. We will show that Q has an amenable direct summand. Since $Q' \cap \tilde{M}^\omega \not\subset M^\omega$, there exist a sequence $x_n \in (\tilde{M})_1$ such that:

- $x_n \in L^2(\tilde{M}) \ominus L^2(M)$, for all $n \in \mathbb{N}$;

- There exists $\varepsilon > 0$ such that $\|x_n\|_2 \geq \epsilon$ for all $n \in \mathbb{N}$;
- $\lim_n \|[u, x_n]\|_2 = 0$ for all $u \in \mathcal{U}(Q)$.
- $x_n = x_n^*$.

Since $x_n \in (\tilde{M})_1$ for all $n \in \mathbb{N}$, the vectors $x_n \in H$ are left and right uniformly bounded, and one can consider the sequence $\xi_n = x_n \otimes_M \cdots \otimes_M x_n \in H^{\otimes_M k}$. One checks that these are almost Q-central vectors, because the x_n's are. Let's show that up to some slight modifications they are Qq-tracial as well, for some $q \in \mathcal{Z}(Q)$.

For all n, define by induction elements $y_i^{(n)} \in M$, $i = 0, \cdots, k$ by $y_0^{(n)} = 1$, $y_{i+1}^{(n)} = E_M(x_n y_i^{(n)} x_n)$. Then an easy computation gives, for all $n \in \mathbb{N}$ and $a \in M$,

$$\langle a\xi_n, \xi_n \rangle = \langle a x_n y_{k-1}^{(n)}, x_n \rangle = \tau(a y_k^{(n)}).$$

Moreover, for all $n \in \mathbb{N}$, $\|x_n\| \leq 1$ implies $\|y_k^{(n)}\| \leq 1$. So taking a subsequence if necessary, one can assume that $(y_k^{(n)})$ converges weakly to some $b \in Q' \cap M_+$.

Claim. $\tau(b) \geq \varepsilon^{2k}$, so that $b \in M$ is a nonzero element.

To prove this claim, first observe that for any $0 \leq i, j \leq k-1$, one has:

$$\begin{aligned}\tau(y_i^{(n)} y_{j+1}^{(n)}) &= \tau(y_i^{(n)} E_M(x_n y_j^{(n)} x_n)) = \tau(y_i^{(n)} x_n y_j^{(n)} x_n) \\ &= \tau(E_M(x_n y_i^{(n)} x_n) y_j^{(n)}) = \tau(y_{i+1}^{(n)} y_j^{(n)}).\end{aligned}$$

Remembering that $k = 2^p$, the relation above and Cauchy-Schwarz inequality give:

$$\begin{aligned}\tau(y_k^{(n)}) = \tau(y_{2^p}^{(n)}) &= \tau(y_{2^{p-1}}^{(n)} y_{2^{p-1}}^{(n)}) \\ &\geq \tau(y_{2^{p-1}}^{(n)})^2 \geq \cdots \geq \tau(y_1^{(n)})^{2^{(p-1)}} \\ &= \tau(x_n^2)^{k/2} \geq \varepsilon^k.\end{aligned}$$

This proves the claim. Therefore there exists $\delta > 0$ such that $q = \chi_{[\delta,\infty[}(E_Q(b)) \neq 0$. Note that $q \in \mathcal{Z}(Q)$ and take $c \in \mathcal{Z}(Q)_+$ such that $q = cE_Q(b)$.
Finally, we get that the sequence $\eta_n = c^{1/2} \cdot \xi_n \in H^{\otimes_M K}$ satisfies:

- (η_n) is almost Qq-tracial: $\forall a \in Qq$, $\lim_n \langle a\eta_n, \eta_n \rangle = \tau(c^{1/2} a c^{1/2} b) = \tau(aq)$.
- (η_n) is almost Q-central.

Therefore as Qq-Qq bimodules, we have:

$$L^2(Qq) \subset_w H^{\otimes_M K} \subset_w L^2(M) \otimes L^2(M) \subset_w L^2(Qq) \otimes L^2(Qq),$$

so that Qq is amenable. $\square$

Proof of Proposition II.2.5. Assume that $\pi^{\otimes k}$ is weakly contained in the regular representation for some $k \geq 1$, and consider a von Neumann subalgebra $Q \subset M$ with no amenable direct summand.

Fix $\varepsilon > 0$. We want to find a $t > 0$ such that for all $s \in [0,t]$,$\|x - \alpha_s(x)\|_2 \leq \varepsilon$ for all x in $(Q' \cap M)_1$.

By Lemma II.2.8, we have $Q' \cap \tilde{M}^\omega \subset M^\omega$ for any free ultrafilter ω on $\mathbb{N}$. It implies that there exists a finite set $x_1, \cdots, x_n \in Q$ and a $\delta > 0$ such that for any $y \in \tilde{M}$ such that $\|yx_i - x_i y\|_2 \leq \delta$ for all $i = 1, \cdots, n$ we have

$$\|y - E_M(y)\|_2 \leq \varepsilon/2.$$

Now let $t > 0$ be such that for any $s \in [0,t]$, and any $i = 1, \cdots, n$, $\|\alpha_s(x_i) - x_i\|_2 \leq \delta/2$. Take $s \in [0,t]$ and $x \in (Q' \cap M)_1$.

We have for all $i = 1, \cdots, n$

$$\begin{aligned}
\|\alpha_s(x)x_i - x_i\alpha_s(x)\|_2 &= \|x\alpha_{-s}(x_i) - \alpha_{-s}(x_i)x\|_2 \\
&\leq 2\|x\|\|\alpha_{-s}(x_i) - x_i\|_2 + \|xx_i - x_i x\|_2 \\
&\leq 2\|x_i - \alpha_s(x_i)\|_2 \\
&\leq \delta.
\end{aligned}$$

By definition of δ, we get

$$\|\alpha_s(x) - E_M(\alpha_s(x))\|_2 \leq \varepsilon/2.$$

By Lemma II.2.3 we conclude that $\|\alpha_{2s}(x) - x\|_2 \leq \varepsilon$, as desired. □

II.2.3 Position of rigid subalgebras of M

Under the assumption that the action is mixing, the following result shows that diffuse, rigid subalgebras of M either lie in the group algebra $L\Gamma$ or their normalizers lie in the Cartan subalgebra A.

Theorem II.2.9. *Assume that $\Gamma \curvearrowright A$ is the Gaussian action associated with a mixing representation of the group Γ. Put $M = A \rtimes \Gamma$, $\tilde{M} = (A \overline{\otimes} A) \rtimes \Gamma$ and define $(\alpha_t) \in \mathrm{Aut}(\tilde{M})$ as in section II.2.1. Let $p \in M$ be a projection, and $Q \subset pMp$ be an (α_t)-rigid subalgebra, in the sense of Definition II.2.2. Denote by $P = \mathcal{QN}_{pMp}(Q)''$.*

Then either $P \prec_M A$, or $Q \prec_M L\Gamma$.

To prove such a rigidity result, we follow a very standard strategy, initiated in [Po06a, Theorem 4.1] and applied to several contexts [Po06d, IPP08, HR11, PS12]. Our proof is a direct adaptation of the proof of [IPV13, Theorem 4.2].

Proof. Assume that no corner of $P := \mathcal{QN}_{pMp}(Q)''$ embeds into A inside M. We will proceed in three steps to show that $Q \prec L\Gamma$.

For subalgebras $Q_1, Q_2 \subset \tilde{M}$, an element $x \in \tilde{M}$ is said to be Q_1-Q_2-*finite* if there exist $x_1, \cdots, x_n \in \tilde{M}$ such that

$$Q_1 x \subset \sum_{i=1}^n x_i Q_2 \text{ and } xQ_2 \subset \sum_{i=1}^n Q_1 x_i.$$

Note that $\mathcal{QN}_{\tilde{M}}(Q_1)$ is exactly the set of Q_1-Q_1 finite elements.

Step 1. For all t small enough, there exists a non-zero Q-$\alpha_t(Q)$-finite element a_t in $p\tilde{M}\alpha_t(p)$.

By assumption, for all t small enough and all $u \in \mathcal{U}(Q)$ we have $\|u - \alpha_t(u)\|_2^2 \leq 1/2$. This implies that

$$\tau(u\alpha_t(u^*)) \geq 3/4 > 0 \text{ for all } u \in \mathcal{U}(Q). \tag{II.5}$$

Denote by $\mathcal{C} \subset p\tilde{M}\alpha_t(p)$ the strong closure of $\mathcal{C}_0 := \mathrm{conv}(\{u\alpha_t(u^*)\,,\, u \in \mathcal{U}(Q)\})$. Since $\mathcal{C}_0$ is in the unit ball of $\tilde{M}$, $\mathcal{C}$ is closed in the norm $\|\cdot\|_2$.

Consider the unique element a_t of $\mathcal{C}$ which minimizes the norm $\|\cdot\|_2$. By uniqueness of a_t, and since $\mathcal{C}$ is invariant under the maps $x \mapsto vx\alpha_t(v^*)$ for all $v \in \mathcal{U}(Q)$, we get that $a_t = va_t\alpha_t(v^*)$ for all $v \in \mathcal{U}(Q)$, and a_t is indeed Q-$\alpha_t(Q)$-finite.

Moreover, by (II.5) we get that $\tau(v_t) \geq 3/4$ and so $a_t \neq 0$.

Step 2. There exists a non-zero element $a_1 \in p\tilde{M}\alpha_1(p)$ which is Q-$\alpha_1(Q)$-finite. In particular $\alpha_1(Q) \prec_{\tilde{M}} M$.

Take a_t as in Step 1, with t of the form $t = 1/2^k$, $k \geq 1$. We will show that there exists $d \in P$ such that $a_{2t} := \alpha_t(\beta(a_t^*)da_t)$ is non-zero. This element $a_{2t} \in p\tilde{M}\alpha_{2t}(p)$ is easily seen to be Q-$\alpha_{2t}(Q)$ finite, so a_{2t} satisfies Step 1 with $2t$ instead of t. Going on inductively this is enough to prove the existence of a_1.

Assume by contradiction that $\beta(a_t^*)da_t = 0$ for all $d \in P$. Denote by $q \in p\tilde{M}p$ the projection onto the closed linear span of $\{\mathrm{range}(da_t)\,,\, d \in P\}$. We see that $\beta(q)q = 0$ and $Q \in P' \cap p\tilde{M}p$.

But if π is mixing and $P \not\prec_M A$, Lemma A.4.5 implies that $P' \cap p\tilde{M}p \subset pMp$. So we have $0 = \beta(q)q = q$ which contradicts the fact that q majorizes the left support of $pa_t = a_t \neq 0$.

Step 3. Conclusion: $Q \prec_M L\Gamma$.

Denote by u_g, $g \in \Gamma$ the canonical unitaries which implement the action of Γ.

Assume by contradiction that $Q \not\prec_M L\Gamma$: there exists a sequence $(w_n) \subset \mathcal{U}(Q)$ such that $\lim_n \|E_{L\Gamma}(xw_ny)\|_2 = 0$ for all $x, y \in M$

We claim that $\lim_n \|E_M(x\alpha_1(w_n)y)\|_2 = 0$ for all $x, y \in \tilde{M}$. By a linearity/density argument, it suffices to prove this equality for $x = (a \otimes b)u_s \in \tilde{M}$ and $y = (c \otimes d)u_t \in \tilde{M}$, with $a, b, c, d \in A$, $s, t \in \Gamma$. Now writing $w_n = \sum_{k\in\Gamma} x_{g,n}u_g$, an easy calculation gives

$$\begin{aligned}
\|E_M(x\alpha_1(w_n)y)\|_2^2 &= \left\| E_M\left(x\sum_g (1 \otimes x_{g,n})u_g y\right)\right\|_2^2 \\
&= \left\| E_M\left(\sum_g \left(a\sigma_{sg}(c) \otimes b\sigma_s(x_{g,n})\sigma_{sg}(d)\right) u_{sgt}\right)\right\|_2^2 \\
&= \sum_g \|a\sigma_{sg}(c)\|_2^2\, |\tau(b\sigma_s(x_{g,n})\sigma_{sg}(d))|^2 \\
&\leq \sum_g \|a\|_\infty^2 \|c\|_2^2\, |\tau(b\sigma_s(x_{g,n})\sigma_{sg}(d))|^2 \\
&= \|a\|_\infty^2 \|c\|_2^2 \|E_{L\Gamma}\left((bu_s)w_nd\right)\|_2^2,
\end{aligned}$$

which tends to 0 when n goes to infinity. This contradicts $\alpha_1(Q) \prec_{\tilde{M}} M$. □

Remark II.2.10. From this proof, we see that the assumption that Q is (α_t)-rigid can be replaced by the following weaker assumption: for t small enough, there exist $z \in \tilde{M}$ and $c > 0$ such that

$$|\tau(\alpha_t(u^*)zu)| \geq c, \text{ for all } u \in \mathcal{U}(Q). \tag{II.6}$$

Assuming that Γ is ICC and using again the mixing property, one can deduce a more accurate result.

Corollary II.2.11. *Consider a mixing Gaussian action $\Gamma \curvearrowright A$ of an ICC group Γ. Define $M = A \rtimes \Gamma$ and $(\alpha_t) \in \mathrm{Aut}(\tilde{M})$ as in section II.2.1. Let $p \in M$ be a projection, and $Q \subset pMp$ be an (α_t)-rigid subalgebra, in the sense of Definition II.2.2. Denote by $P = \mathcal{QN}_{pMp}(Q)''$.*

Then at least one of the following assertions occurs.

1. *$Q \prec_M 1$;*
2. *$P \prec_M A$;*
3. *There exists a unitary $v \in M$ such that $v^*Pv \subset L\Gamma$.*

Proof. Note that for all $r \in Q' \cap pMp$, the subalgebra $rQ \subset rMr$ is (α_t)-rigid. So if $P \not\prec A$, Theorem II.2.9 applied to all such rQ's implies that for all $r \in Q' \cap pMp$, $rQ \prec L\Gamma$. Now one can apply Proposition A.4.6.3 because the inclusion $L\Gamma \subset M$ is mixing (relative to $\mathbb{C}$). This implies that either 1 or 3 (or both) holds true. $\square$

We now provide a new variant of Theorem II.2.9, which relies on spectral hypotheses on π instead of mixing hypotheses. Note however that if Γ is non-amenable, the assumption that $\pi^{\otimes k}$ is weakly contained in the regular representation implies that π is weakly mixing.

Theorem II.2.12. *Assume that $\Gamma \curvearrowright A$ is the Gaussian action associated with a representation π such that $\pi^{\otimes k}$ is weakly contained in the regular representation for some $k \geq 1$. Put $M = A \rtimes \Gamma$, $\tilde{M} = (A \overline{\otimes} A) \rtimes \Gamma$ and define $(\alpha_t) \in \mathrm{Aut}(\tilde{M})$ as in section II.2.1. Let $p \in M$ be a projection, and $Q \subset pMp$ be an (α_t)-rigid subalgebra. Denote by $P = \mathcal{QN}_{pMp}(Q)''$.*

Then either P has an amenable direct summand or $Q \prec_M L\Gamma$.

Proof. Assume that $Q \subset pMp$ is an (α_t)-rigid subalgebra and that $P := \mathcal{QN}_{pMp}(Q)''$ has no amenable direct summand.

Denote by $P_0 = P \oplus (1-p)M(1-p)$. Then P_0 has no amenable direct summand. Using Lemma II.2.8 we have that $P_0' \cap \tilde{M} \subset M$. In particular $P' \cap p\tilde{M}p \subset pMp$.

Now we can repeat Steps 1-3 of the proof of Theorem II.2.9 word by word to get $Q \prec_M L\Gamma$. $\square$

From Theorem II.2.12 we can deduce the following relative solidity result. The first result of this type is due to Ozawa [BO08, Theorem 15.3.10] (inspired from [Oz04]) and deals with (some) plain Bernoulli shifts[1]. Then Chifan and Ioana [CI10] managed to weaken the mixing assumption, and proved the result for generalized Bernoulli shifts associated with actions with amenable stabilizers. The result that we give here is an improved version of [Bo12, Proposition 4.1].

Corollary II.2.13. *Assume that $\Gamma \curvearrowright A$ is the Gaussian action associated with a representation π such that $\pi^{\otimes k}$ is weakly contained in the regular representation for some $k \geq 1$. Put $M = A \rtimes \Gamma$ and take a projection $p \in M$.*

For any $Q \subset pMp$ such that $Q \not\prec_M L\Gamma$, we have that $Q' \cap pMp$ is amenable.

[1] Ozawa's approach is totally different from what is presented here.

Proof. Assume by contradiction that $Q \not\prec_M L\Gamma$ and that $P := Q' \cap pMp$ is not amenable. Then we can find a central projection $z \in \mathcal{Z}(P)$ such that Pz has no amenable direct summand. By Corollary II.2.6, we get that $(Pz)' \cap zMz$ is (α_t)-rigid. But since $Qz \subset (Pz)' \cap zMz$, we deduce that Qz is (α_t)-rigid.

Since $Q \not\prec_M L\Gamma$, we also have that $Qz \not\prec_M L\Gamma$. Applying Theorem II.2.12 to Qz, we get that the quasi-normalizer of Qz inside zMz has an amenable direct summand. But this quasi-normalizer contains (unitaly) Pz, which has no amenable direct summand by definition of z. This is impossible. □

This corollary is interesting on its own, but it also has an application regarding *solid ergodicity* of Gaussian actions.

II.2.4 Application: Gaussian actions and solid ergodicity

Definition II.2.14. A measure preserving equivalence relation $\mathcal{R}$ on a probability space (X, μ) is called *solidly ergodic* if for any subrelation $\mathcal{S}$ of $\mathcal{R}$, there exists a countable measurable partition $(X_n)_{n\geq 0}$ of X into measurable $\mathcal{S}$-invariant subsets with:

- $\mathcal{S}_{|X_0}$ hyperfinite ;
- $\mathcal{S}_{|X_n}$ is strongly ergodic for all $n \geq 1$."

Solid ergodicity is a very strong property because it gives valuable information on the ergodic decomposition of *any* sub-equivalence relation.

The name *solid ergodicity* was introduced by Gaboriau [Ga10, definition 5.4] and is due to the following characterisation found by Chifan and Ioana [CI10].

Proposition II.2.15 ([CI10], Proposition 6)**.** *A measure preserving equivalence relation $\mathcal{R}$ on a standard probability space (X, μ) is solidly ergodic if and only if for any diffuse von Neumann subalgebra Q of $L^\infty(X, \mu)$, the relative commutant $Q' \cap L\mathcal{R}$ is amenable.*

Chifan and Ioana applied this criterion to provide the first example of solidly ergodic action: if $\Gamma \curvearrowright I$ is an action of a group on a countable set I, with amenable stabilizers then the orbit equivalence relation of the associated Bernoulli action is solidly ergodic.

Together with Ozawa's work [Oz09], Proposition II.2.15 also implies that the orbit equivalence relation induced by $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}^2$ is solidly ergodic.

Combining Proposition II.2.15 with Proposition II.2.13, we obtain solid ergodicity results for a large class of Gaussian actions.

Corollary II.2.16. *Assume that $\pi : \Gamma \to \mathcal{O}(H_\mathbb{R})$ is an orthogonal representation such that $\pi^{\otimes k}$ is weakly contained in the regular representation for some $k \geq 1$.*

Then the orbit equivalence relation associated with the Gaussian action σ_π is solidly ergodic.

Although this result might sound slightly more general than [Bo12, Theorem A], we were not able to provide new examples of solidly ergodic Gaussian actions. Moreover note that for quasi-regular representations, this theorem is equivalent to [Bo12, Theorem A].

II.2.5 Deformation/rigidity in the ultraproduct algebra

In this section we apply deformation/rigidity techniques in the ultraproduct algebra. This idea was first used in [Pe09]. Our main result is Theorem II.2.19, which will play a crucial role in Section II.3.

Let $\sigma_\pi : \Gamma \curvearrowright A$ be the Gaussian action associated with an orthogonal representation π. As in Section II.2.1, denote by $M = A \rtimes \Gamma$, $\tilde{M} = (A \overline{\otimes} A) \rtimes \Gamma$ and by $((\alpha_t)_t, \beta)$ the corresponding s-malleable deformation of M.

Let ω be a free ultrafilter on $\mathbb{N}$. The malleable deformation $(\alpha_t)_t$ of M induces a one parameter group of automorphisms, also denoted by $(\alpha_t)_t$, of the ultraproduct algebra $(\tilde{M})^\omega$ by the formula

$$\alpha_t(x) := (\alpha_t(x_n))_n, \text{ for all } x = (x_n) \in (\tilde{M})^\omega, t \in \mathbb{R}.$$

Definition II.2.17. For a given $x = (x_n) \in (\tilde{M})^\omega$, the map $t \mapsto \alpha_t(x)$ may not be continuous for the norm $\|\cdot\|_2$. Whenever it is continuous, we will say that x is (α_t)-*rigid*.

A sufficient condition for an element $x = (x_n)$ to be (α_t)-rigid is that the deformation converges uniformly to the identity on the set $\{x_n\, , \, n \in \mathbb{N}\}$. Since the deformation is trivial on $L\Gamma$, any element of $(L\Gamma)^\omega$ is (α_t)-rigid.

The spectral gap argument also yields rigidity in M^ω.

Lemma II.2.18. *Let $p \in M$ be a projection. If $Q \subset pMp$ has no amenable direct summand, then the deformation converges pointwise (even uniformly) to the identity in the norm $\|\cdot\|_2$ on the unit ball $(Q' \cap M^\omega)_1$.*

Proof. The proof is exactly the same as the one of Proposition II.2.5. □

The following result shows that for if π is mixing one can transfer the rigidity of an element $x \in M^\omega$ to its relative commutant in M.

Theorem II.2.19. *Let $\Gamma \curvearrowright A$ be a mixing Gaussian action. Put $M = A \rtimes \Gamma$. Assume that $x = (x_n) \in M^\omega$ is an (α_t)-rigid element and consider a subalgebra $Q \subset M$ of elements of M that commute (inside M^ω) with x. Put $P = \mathcal{QN}_M(Q)''$.*

If $x \notin A^\omega \rtimes \Gamma$, then $P \prec_M A$ or $Q \prec L\Gamma$.

Proof. Assume that $x \notin (A^\omega \rtimes \Gamma)$. If we could show that Q is (α_t)-rigid then we would conclude with Theorem II.2.9. We will rather show that Q satisfies condition II.6 and apply Remark II.2.10.

Define $y = x - E_{A^\omega \rtimes \Gamma}(x) \neq 0$ and write $y = (y_n)$. Dividing y if necessary by $\|y\|_2$, one can assume that $\|y\|_2 \leq 1$. For all n write $y_n = \sum_g y_{n,g} u_g$, where u_g, $g \in \Gamma$ denote the unitaries of M implementing the action of Γ. One checks that:

- y is (α_t)-rigid, since α_t commutes with $E_{\tilde{A}^\omega \rtimes \Gamma}$ for all t;
- $yd = dy$, for any $d \in Q$;
- For all g, y is orthogonal to $(y_{n,g} u_g)_n \in A^\omega \rtimes \Gamma$, so that $\lim_{n \to \omega} \|y_{n,g}\|_2 = 0$.

Using Lemma A.1.4, this last condition implies that

$$\lim_{n\to\omega}\langle y_n\xi y_n^*,\eta\rangle = 0, \forall \xi,\eta \in L^2(\tilde{M})\ominus L^2(M). \tag{II.7}$$

Fix $\varepsilon > 0$. Then there exists $t_0 > 0$ such that for all $t \in [0,t_0]$ we have $\|\alpha_t(y) - y\|_2 < \varepsilon$.
Let $t \in [0,t_0]$ and $u \in \mathcal{U}(Q)$. For $a \in M$ define $\delta_t(a) = \alpha_t(a) - E_M \circ \alpha_t(a) \in L^2(\tilde{M})\ominus L^2(M)$. We have

$$\begin{aligned}\lim_{n\to\omega}\|\delta_t(u)y_n - \delta_t(uy_n)\|_2 &= \lim_{n\to\omega}\|(1-E_M)(\alpha_t(u)y_n - \alpha_t(uy_n))\|_2\\ &\leq \lim_{n\to\omega}\|\alpha_t(u)y_n - \alpha_t(u)\alpha_t(y_n)\|_2\\ &= \|y-\alpha_t(y)\|_2 \leq \varepsilon.\end{aligned}$$

Similarly $\lim_{n\to\omega}\|y_n\delta_t(u) - \delta_t(y_nu)\|_2 < \varepsilon$. Hence we get

$$\begin{aligned}\lim_{n\to\omega}\|\delta_t(u)y_n\|_2^2 &\leq \lim_{n\to\omega}\langle\delta_t(uy_n),\delta_t(u)y_n\rangle + \varepsilon\\ &= \lim_{n\to\omega}\langle\delta_t(y_nu),\delta_t(u)y_n\rangle + \varepsilon\\ &\leq \lim_{n\to\omega}\langle y_n\delta_t(u)y_n^*,\delta_t(u)\rangle + 2\varepsilon.\end{aligned}$$

With (II.7), we obtain

$$\|\delta_t(u)y\|_2^2 \leq 2\varepsilon. \tag{II.8}$$

But exactly as in the proof of Popa's transversality lemma (Lemma II.2.3), we show that

$$\begin{aligned}\|\alpha_{2t}(u)y - uy\|_2 &\leq \|\alpha_t(u)y - \alpha_{-t}(u)y\|_2 + 2\|y-\alpha_t(y)\|_2\\ &\leq 2\|\delta_t(u)y\|_2 + 2\varepsilon\end{aligned}$$

With (II.8) we thus obtain

$$\|\alpha_{2t}(u)y - uy\|_2 < 2\sqrt{2\varepsilon} + 2\varepsilon \leq 6\sqrt{\varepsilon},$$

if we assume that $\varepsilon < 1$. Put $z = E_M(yy^*) \neq 0$ (conditionnal expectation inside M^ω, onto M). We have

$$\|\alpha_{2t}(u)y - uy\|_2^2 = 2\|y\|_2^2 - 2\Re(\tau(\alpha_{2t}(u^*)zu)) < 36\varepsilon.$$

If ε was chosen to be smaller than $\|y\|_2^2/18$, then $c := \|y\|_2^2 - 18\varepsilon$ is positive and satisfies

$$|\tau(\alpha_{2t}(u^*)zu)| \geq c.$$

This inequality is true for all $t \in [0,t_0]$ and all $u \in \mathcal{U}(Q)$, so we can apply Remark II.2.10 to get the result. □

II.2.6 Application: structural properties of M

From Sections II.2.3 and II.2.5, we can deduce the following result about primeness and property Gamma.

Theorem II.2.20. *Assume that $\Gamma \curvearrowright A$ is the Gaussian action associated with a mixing representation π such that $\pi^{\otimes k}$ is weakly contained in the regular representation for some $k \geq 1$. Put $M = A \rtimes \Gamma$.*

Let $Q \subset M$ be a von Neumann subalgebra such that $Q \not\prec L\Gamma$. Then there exists countably many projections $(p_n)_{n\geq 0}$ in $\mathcal{Z}(Q)$ such that $\sum_{n\geq 0} p_n = 1$ and

- *p_0Q is amenable;*
- *For all $n \geq 1$, p_nQ is prime and does not have property Gamma.*

Proof. We proceed in two steps.

STEP 1. Construction of the projections p_n.

Naturally, take for p_0 the maximal projection in $\mathcal{Z}(Q)$ such that p_0Q is amenable. Let us show that $(1-p_0)\mathcal{Z}(Q)$ is discrete.

Otherwise one can find a projection $p \in \mathcal{Z}(Q)$ with $p \leq 1-p_0$ such that $p\mathcal{Z}(Q)$ is diffuse. Since the action is mixing and $pQ \not\prec L\Gamma$, Proposition A.4.6.1 implies that $p\mathcal{Z}(Q) \not\prec L\Gamma$. Therefore Corollary II.2.13 implies that pQ is amenable, which contradicts the fact that $p \leq 1-p_0$.

Thus we obtain (at most) countably many projections $(p_n)_{n\geq 0}$ such that p_0Q is hyperfinite, and p_nQ is a non-hyperfinite factor for all $n \geq 1$.

STEP 2. For any $n \geq 1$, p_nQ does not have property Gamma and is prime.

We have to show that for every projection $p \in M$, any non-hyperfinite subfactor $N \subset pMp$ such that $N \not\prec_M L\Gamma$ is non-Gamma and prime.

Primeness. If $N = N_1 \overline{\otimes} N_2$, then N_1 and N_2 are factors, and one of them, say N_1, is non-amenable. Hence Corollary II.2.13 implies that $N_2 \prec_M L\Gamma$. By Proposition A.4.6.1, either N_2 is discrete or $N \prec_M L\Gamma$. The only possible case is that N_2 is discrete. So N is prime.

Non Property Gamma. Fix a free ultrafilter ω on $\mathbb{N}$. Put $B = N' \cap N^\omega$.

Since $N \subset pMp$ has no amenable direct summand, Lemma II.2.18 and Theorem II.2.19 imply that either $N' \cap (pMp)^\omega \subset A^\omega \rtimes \Gamma$, or $N \prec_M A$ or $N \prec_M L\Gamma$. But the last two possibilities are clearly excluded. We deduce that $B \subset A^\omega \rtimes \Gamma$.

Assume by contradiction that the II$_1$-factor N has property Gamma, *i.e.* that B is diffuse. The proof of [Oz04, Proposition 7], shows that there exists a sequence of τ-independent commuting projections $p_n \in N$ of trace 1/2, such that $(p_n) \in N' \cap N^\omega$, and if $C = \{p_n \,|\, n \in \mathbb{N}\}''$, then $C' \cap N$ is not amenable.

By Corollary II.2.13, we get that $C \prec_M L\Gamma$.

At this point, remark that the sequence of unitaries $w_n = 2p_n - 1 \in \mathcal{U}(C)$ converges weakly to 0, and that $(w_n) \in N' \cap N^\omega \subset A^\omega \rtimes \Gamma$. The following claim leads to a contradiction.

Claim. For all $x, y \in M, \lim_n \|E_{L\Gamma}(xw_ny)\|_2 = 0$.

Denote by $u_g \in M$, $g \in \Gamma$ the unitaries implementing the action of Γ. By linearity and density, it suffices to prove the claim for $x = au_h$, $y = bu_k$, for $a, b \in A$, $h, k \in \Gamma$. Write $w_n = \sum_{g\in\Gamma} a_{n,g}u_g$ and let $\varepsilon > 0$. Since $(w_n) \in A^\omega \rtimes \Gamma$, there exists $F \in \Gamma$ finite such that

$$\|P_F(w_n) - w_n\|_2 < \frac{\varepsilon}{2\|a\|\|b\|}, \ \forall n \in \mathbb{N}.$$

Now we have:

$$\begin{aligned}\|E_{L\Gamma}(xP_F(w_n)y)\|_2^2 &= \sum_{g\in F} |\tau(a\sigma_h(a_{n,g})\sigma_{hg}(b))|^2 \\ &= \sum_{g\in F} |\tau(\sigma_{h^{-1}}(a)w_nu_g^*\sigma_g(b))|^2.\end{aligned}$$

This quantity can be made smaller than $\varepsilon^2/4$ for n large enough, and we get that $\|E_{L\Gamma}(xw_ny)\|_2 < \varepsilon$ for n large enough. That proves the claim and gives the desired contradiction. □

Remark II.2.21. Theorem II.2.20 is a von Neumann algebraic analogue of Theorem II.2.16. However we added a mixing assumption in order to prove it. This mixing assumption can be weakened to a relative mixing assumption though, see [Bo12, Theorem B].

II.3 Position of well-normalized subalgebras

We continue our analysis of remarkable von Neumann subalgebras of the crossed product von Neumann algebra by a mixing Gaussian action. The whole section will be devoted to proving Theorem II.3.1 which generalizes [Po06b, Theorem 4.2].

II.3.1 Statement of the main result

Theorem II.3.1. *Consider a mixing Gaussian action $\Gamma \curvearrowright (X, \mu)$ of a discrete countable group Γ. Put $A = L^\infty(X, \mu)$, and $M = A \rtimes \Gamma$. Assume that $B \subset M$ is an abelian subalgebra which is normalized by a sequence of unitaries $v_n \in \mathcal{U}(L\Gamma)$ such that $v_n \to 0$ weakly.*

Then $B' \cap M \prec_M A$ or $B \prec_M L\Gamma$.

For later use, let us mention a tensor product version of this result which we proved in [Bo13]. We also allow amplifications. The proof does not change much compared to that of Theorem II.3.1. It is a generalization of [Io11, Theorem 6.1] and [IPV13, Theorem 5.1], in the mixing case.

Theorem II.3.2. *For $i = 1, 2$, consider mixing Gaussian actions $\Gamma_i \curvearrowright A_i$ of discrete countable groups Γ_i, and put $M_i = A_i \rtimes \Gamma_i$, $A = A_1 \mathbin{\overline{\otimes}} A_2$, $\Gamma = \Gamma_1 \times \Gamma_2$ and*

$$M = M_1 \mathbin{\overline{\otimes}} M_2 = A \rtimes \Gamma.$$

Let $t > 0$. Realize $(L\Gamma)^t \subset M^t$ by fixing an integer $n \geq t$ and a projection $p \in L\Gamma \otimes M_n(\mathbb{C})$ with trace t/n. Let $D \subset M^t$ be an abelian von Neumann subalgebra, and denote by Λ'' the von Neumann algebra generated by the group of unitaries $\Lambda = \mathcal{N}_{M^t}(D) \cap \mathcal{U}((L\Gamma)^t)$. Make the following assumptions:

- *(i) $\Lambda'' \nprec_M L\Gamma_1 \otimes 1$ and $\Lambda'' \nprec_M 1 \otimes L\Gamma_2$;*
- *(ii) $D \nprec L\Gamma_1 \otimes M_2$ and $D \nprec_M M_1 \otimes L\Gamma_2$.*

Denote by $C = D' \cap M^t$. Then for all projections $q \in \mathcal{Z}(C)$, $Cq \prec_M A$.

Bernoulli actions $\Gamma \curvearrowright A = A_0^{\otimes\Gamma}$ have a strong algebraic structure, given by the cylinders $A_0^{\otimes F}$, $F \subset \Gamma$ finite. This structure was used via the so-called *clustering property* (see [Po06b, Section 1,2,3]), one of the main ingredients to prove the Bernoulli analogue of Theorem II.3.1, namely [Po06b, Theorem 4.2].

General mixing Gaussian actions do not have such an algebraic structure, but they are 2-mixing. This 2-mixing property will allow to replace cylinders by general finite dimensional subspaces of A.

II.3.2 2-mixing property

Definition II.3.3. A trace-preserving action $\Gamma \curvearrowright^\sigma A$ of a countable group on an abelian von Neumann algebra is said to be 2-*mixing* if for any $a, b, c \in A$, the quantity $\tau(a\sigma_g(b)\sigma_h(c))$ tends to $\tau(a)\tau(b)\tau(c)$ as $g, h, g^{-1}h$ tend to infinity.

Proposition II.3.4. *An action $\Gamma \curvearrowright^\sigma A$ is 2-mixing if and only if for all $a, b, c \in A$, one has*

$$|\tau(a\sigma_g(b)\sigma_h(c)) - \tau(a)\tau(\sigma_g(b)\sigma_h(c))| \to 0,$$

when $g \to \infty$, $h \to \infty$.

Proof. The *if* part is straightforward. For the converse, assume that σ is 2-mixing. It is sufficient to show that if $a, b, c \in A$, with $\tau(a) = 0$, then $\tau(a\sigma_g(b)\sigma_h(c)) \to 0$, as $g, h \to \infty$.

Assume by contradiction that there exist sequences $g_n, h_n \in \Gamma$ going to infinity, and $\delta > 0$ such that $|\tau(a\sigma_{g_n}(b)\sigma_{h_n}(c))| \geq \delta$, for all n. Then two cases are possible:

Case 1. The sequence $g_n^{-1}h_n$ is contained in a finite set. Then taking a subsequence if necessary, one can assume that $g_n^{-1}h_n = k$ is constant. Then for all n, we get

$$\tau(a\sigma_{g_n}(b)\sigma_{h_n}(c)) = \tau(a\sigma_{g_n}(b\sigma_k(c)).$$

But since σ is mixing this quantity tends to 0 as n tends to infinity.

Case 2. The sequence $g_n^{-1}h_n$ is not contained in a finite set. Then taking a subsequence if necessary, one can assume that $g_n^{-1}h_n \to \infty$ when $n \to \infty$. Then the 2-mixing implies that $\tau(a\sigma_{g_n}(b)\sigma_{h_n}(c)) \to 0$.

In both cases, we get a contradiction. $\square$

Of course any 2-mixing action is mixing. The converse holds for Gaussian actions.

Proposition II.3.5. *If $\Gamma \curvearrowright^\sigma A$ is the Gaussian action associated with a mixing representation π on H, then σ is 2-mixing.*

Proof. By a linearity/density argument, it is enough to prove that for all $\xi, \eta, \delta \in H$, and all sequences $g_n, h_n \in \Gamma$ tending to infinity, one has

$$\lim_n [\tau(w(\xi)\sigma_{g_n}(w(\eta))\sigma_{h_n}(w(\delta))) - \tau(w(\xi))\tau(\sigma_{g_n}(w(\eta))\sigma_{h_n}(w(\delta)))] = 0,$$

where we used the notations of Definition II.1.5 for $w(\xi), w(\eta)$ and $w(\delta)$. But we see that:

- $\tau(\omega(\xi)\sigma_{g_n}(\omega(\eta))\sigma_{h_n}(\omega(\delta))) = \exp(-\|\xi + \pi(g_n)\eta + \pi(h_n)\delta\|^2)$;
- $\tau(\omega(\xi))\tau(\sigma_{g_n}(\omega(\eta))\sigma_{h_n}(\omega(\delta))) = \exp(-\|\xi\|^2 - \|\pi(g_n)\eta + \pi(h_n)\delta\|^2)$.

The difference is easily seen to tend to 0. $\square$

II.3.3 Proof of Theorem II.3.1

Let $B \subset M$ be an abelian von Neumann subalgebra which is normalized by unitaries $v_n \in L\Gamma$ such that the sequence (v_n) converges weakly to 0. Assume that $B \not\prec_M L\Gamma$ and that $C := B' \cap M \not\prec_M A$.

We will proceed in two steps to get a contradiction. First we will collect properties regarding the sequence (v_n) or sequences of the form $(v_n a v_n^*)$, $a \in B$. Then we will derive a contradiction. Before moving on to these two steps, we introduce some notations:

- We denote by $u_g, g \in \Gamma$ the canonical unitaries in M implementing the action of Γ;
- For any element $x \in M$, we denote by $x = \sum_{g\in\Gamma} x_g u_g$ ($x_g \in A$ for all $g \in \Gamma$) its Fourier decomposition.
- If $S \subset \Gamma$ is any subset, denote by $P_S : L^2(M) \to L^2(M)$ the projection onto the closed linear span of the vectors au_g, $a \in A$, $g \in S$.
- If $K \subset A$ is a closed subspace, we denote by $Q_K : L^2(M) \to L^2(M)$ the projection onto the closed linear span of the vectors au_g, $a \in K$, $g \in \Gamma$.

Step 1: Properties of the sequences $(v_n a v_n^*)$, $a \in D$

Lemma II.3.6. *For any free ultrafilter ω on $\mathbb{N}$, and any $a \in B$, the element $(v_n a v_n^*)_n \in M^\omega$ belongs to $A^\omega \rtimes \Gamma$.*

Proof. Let $a \in B$. Since $v_n \in L\Gamma$, the sequence $(v_n a v_n^*)_n$ is (α_t)-rigid in the sense of Definition II.2.17 (we use the notations of Section II.2.1). Assume that this sequence does not belong to $A^\omega \rtimes \Gamma$. Since $B \subset M$ commutes with the sequence $(v_n a v_n^*)_n$, Theorem II.2.19 implies that either $B \prec_M L\Gamma$ or $\mathcal{QN}_M(B)'' \prec_M A$. But $C \subset \mathcal{QN}_M(B)''$, so both possibilities are excluded. Hence $(v_n a v_n^*)_n \in A^\omega \rtimes \Gamma$. $\square$

For an element $x \in M = L\Gamma$, denote by $h(x)$ the height of x: $h(x) = \sup_{g\in\Gamma} |x_g|$, where $x = \sum x_g u_g$ is the Fourier decomposition of x.

Lemma II.3.7. *There exists $\delta > 0$ such that $\liminf_n h(v_n) > \delta$.*

Proof. Assume that the result is false. Taking a subsequence if necessary, we get that $h(v_n) \to 0$. Then we claim that for all finite subset $S \subset \Gamma$, for all $a \in M \ominus L\Gamma$,

$$\|P_S(v_n a v_n^*)\|_2 \to 0.$$

To prove this claim, it is sufficient to show that for any sequence of unitaries $w_n \in \mathcal{U}(L\Gamma)$, and all $a \in A \ominus \mathbb{C}1$,

$$\|E_A(v_n a w_n)\|_2 \to 0.$$

Thus write $v_n = \sum_{g\in\Gamma} v_{n,g} u_g$ and $w_n = \sum_{h\in\Gamma} w_{n,h} u_h$. We get

$$E_A(v_n a w_n) = \sum_g v_{n,g} \sigma_g(a) w_{n,g^{-1}}.$$

Fix $\varepsilon > 0$, and take $F \subset \Gamma$ finite such that $|\tau(\sigma_g(a)\sigma_h(a^*))| < \varepsilon$, as soon as $g^{-1}h \notin F$. A computation gives :

$$\begin{aligned}
\|E_A(v_n a w_n)\|_2^2 &= \sum_{g,h\in\Gamma} v_{n,g} w_{n,g^{-1}} \bar{v}_{n,h} \bar{w}_{n,h^{-1}} \tau(\sigma_g(a)\sigma_h(a^*)) \\
&\leq \sum_{g\in\Gamma}\sum_{h\in gF} v_{n,g} w_{n,g^{-1}} \bar{v}_{n,h} \bar{w}_{n,h^{-1}} \tau(\sigma_g(a)\sigma_h(a^*)) + \varepsilon(\sum |v_{n,g} w_{n,g^{-1}}|)^2 \\
&\leq \sum_{g\in\Gamma} |v_{n,g} w_{n,g^{-1}}| \sum_{h\in gF} h(v_n)\|a\|_2^2 + \varepsilon \\
&\leq |F| h(v_n)\|a\|_2^2 + \varepsilon.
\end{aligned}$$

Which is smaller than 2ε, for n large enough. That proves the claim.
Now since $B \not\prec L\Gamma$, there exists $a \in \mathcal{U}(B)$ with $\|E_{L\Gamma}(a)\|_2 \leq 1/3$. For any $S \subset \Gamma$ finite, we get :

$$\|P_S(v_n a v_n^*)\|_2 \leq \|P_S(v_n(a - E_{L\Gamma}(a))v_n^*)\|_2 + 1/3 \leq 2/3,$$

for n large enough, by the claim. This contradicts the fact that $(v_n a v_n^*) \in A^\omega \rtimes \Gamma$. □

We end this paragraph by a lemma that localizes the Fourier coefficients of elements $v_n a v_n^*$ inside A, for a particular (fixed) $a \in D$. In fact, this lemma will be the starting point of our reasoning by contradiction in Step 2 below, being the initialization of an induction process.

Lemma II.3.8. *There exists an $a \in \mathcal{U}(B)$, a $\delta_0 > 0$, a finite dimensional subspace $K \subset A \ominus \mathbb{C}1$ and a sequence $g_n \in \Gamma$ which converges to infinity, such that :*

$$\liminf_n \|v_n a v_n^* - Q_{\sigma_{g_n}(K)}(v_n a v_n^*)\|_2 < \sqrt{1-\delta_0^2}.$$

Proof. Put $\delta_1 = \liminf h(v_n) > 0$, and consider for all n, $g_n \in \Gamma$ such that $|v_{n,g_n}| = h(v_n)$. Since (v_n) converges weakly to 0, the sequence (g_n) goes to infinity with n. Moreover, one checks that

$$\limsup_n \|v_n - v_{n,g_n} u_{g_n}\|_2 = \sqrt{1-\delta_1^2}.$$

Take $\varepsilon > 0$ small enough so that $\sqrt{1-\delta_1^2} + \varepsilon < 1$, and consider $a \in \mathcal{U}(B)$ such that $\|E_{L\Gamma}(a)\|_2 < \varepsilon$.
Then one can find a finite dimensional space $K \subset A \ominus \mathbb{C}1$, such that $\|a - Q_K(a)\|_2 < \varepsilon$. Finally, we get that $v_n a v_n^*$ is at distance at most $\sqrt{1-\delta_1^2} + \varepsilon$ of $v_{n,g_n} u_{g_n} Q_K(a) v_n^*$, which belongs to the image of the projection $Q_{\sigma_{g_n}(K)}$. Then we get the result with $\delta_0 > 0$ defined by $1 - \delta_0^2 = (\sqrt{1-\delta_1^2} + \varepsilon)^2$. □

Step 2 : Derive a contradiction

Notation. For a finite subset $F \subset \Gamma$, finite dimensional subspaces $K_1, K_2 \subset A$ and $\lambda > 0$, define

$$[K_1 \times \sigma_F(K_2)]^\lambda = \text{conv}\{\lambda a \sigma_g(b) \,|\, a \in K_1, b \in K_2, g \in F, \|a\|_2 \leq 1, \|b\|_2 \leq 1\}.$$

We have that $[K_1 \times \sigma_F(K_2)]^\lambda$ is a closed convex subset $\mathcal{C}$ of A (being the convex hull of a compact subset in a finite dimensional vector space). Then the set $\tilde{\mathcal{C}}$ consisting of vectors $\xi \in L^2(M)$ whose Fourier coefficients $\xi_g = \langle \xi, u_g \rangle$ $(g \in \Gamma)$ belong to $\mathcal{C}$ is a closed convex subset of $L^2(M)$. Hence one can define the orthogonal projection onto this set $Q_{\mathcal{C}} : L^2(M) \to L^2(M)$ as follows. For $x \in L^2(M)$, $Q_{\mathcal{C}}(x)$ is the unique point of $\tilde{\mathcal{C}}$ such that

$$\|x - Q_{\mathcal{C}}(x)\| = \inf_{y \in \tilde{\mathcal{C}}} \|x - y\|.$$

Note that the restriction of $Q_{\mathcal{C}}$ to $L^2(A)$ is equal to the orthogonal projection onto $\mathcal{C}$, and that $Q_{\mathcal{C}}(\sum_{g\in\Gamma} x_g u_g) = \sum_{g\in\Gamma} Q_{\mathcal{C}}(x_g) u_g$.

Remark II.3.9. This notation is consistent with the previous notation Q_K: If $K \subset A$ is a finite dimensional subspace, then $Q_K(a) = Q_{\mathcal{C}}(a)$, where $\mathcal{C} = [\mathbb{C}1 \times \sigma_{\{e\}}(K)]^\lambda$ as soon as $\lambda \geq \|a\|_2$.

Before getting into the heart of the proof, we check some easy properties of these convex sets.

Lemma II.3.10. *Fix $\lambda > 0$ and finite dimensional subspaces $K_1, K_2 \subset A$. Then there exists a constant $\kappa > 0$ such that for all finite $F \subset \Gamma$, and all $x \in [K_1 \times \sigma_F(K_2)]^\lambda$,*

$$\|x\|_\infty \leq \kappa.$$

Proof. Since K_1 and K_2 are finite dimensional, there exists a constant $c > 0$ such that $\|a\|_\infty \leq c\|a\|_2$ for all $a \in K_1$ or $a \in K_2$. One sees that $\kappa = \lambda c^2$ satisfies the conclusion of the lemma. □

Lemma II.3.11. *For finite subsets $F, F' \subset \Gamma$, and finite dimensional subspaces $K_1, K_2, K_1', K_2' \subset A$ and $\lambda, \lambda' > 0$, we have*

$$[K_1 \times \sigma_F(K_2)]^\lambda + [K_1' \times \sigma_{F'}(K_2')]^{\lambda'} \subset [(K_1 + K_1') \times \sigma_{F\cup F'}(K_2 + K_2')]^{\lambda+\lambda'}.$$

Proof. This is straightforward. □

The following lemma is the key of the proof, and yields the contradiction we are after. Indeed, using Lemma II.3.8, and iterating Lemma II.3.12 enough times, we get the absurd statement that there exist unitaries $a_n = v_n a v_n^*$ and elements b_n of the form $Q_{\mathcal{C}_n}(a_n)$ such that $\limsup_n \|a_n - b_n\|_2^2$ is negative.

Lemma II.3.12. *Fix $a \in \mathcal{U}(B)$ and put $a_n = v_n a v_n^*$ for all n. Assume that there exists a sequence of finite subsets $F_n \subset \Gamma$, finite dimensional subspaces $K_1 \subset A$, $K_2 \subset A \ominus \mathbb{C}1$, $\lambda > 0$ and $\delta > 0$ such that:*

- $\sup_n |F_n| < \infty$ *and* $F_n \to \infty$ *(meaning that for all $g \in \Gamma_1$, $g \notin F_n$ for n large enough);*
- $\limsup_n \|a_n - Q_{\mathcal{C}_n}(a_n)\|_2^2 < \|p\|_2^2 - \delta^2$, *where* $\mathcal{C}_n = [K_1 \times \sigma_{F_n}(K_2)]^\lambda$.

Then there exists a sequence of finite subsets $F_n' \subset \Gamma$, finite dimensional subspaces $K_1' \subset A$, $K_2' \subset A \ominus \mathbb{C}1$, and $\lambda' > 0$ such that:

- $\sup_n |F_n'| < \infty$ *and* $F_n' \to \infty$*;*
- $\limsup_n \|a_n - Q_{\mathcal{C}_n'}(a_n)\|_2^2 < \|p\|_2^2 - 3\delta^2/2$, *where* $\mathcal{C}_n' = [K_1' \times \sigma_{F_n'}(K_2')]^{\lambda'}$.

Proof. Let a, a_n, F_n, K_1, K_2, λ, δ and $\mathcal{C}_n$ be as in the lemma. Fix $\varepsilon > 0$, with $\varepsilon \ll \delta$. By Lemma II.3.6 one can find $S \subset \Gamma$ finite such that $\|a_n - P_S(a_n)\|_2 \leq \varepsilon$, for all n. Hence we get that $\limsup_n \|a_n - P_S \circ Q_{\mathcal{C}_n}(a_n)\|_2 < \sqrt{\|p\|_2^2 - \delta^2} + \varepsilon$.

Now following Ioana's idea (see the proof of [Io11, Theorem 5.2] and also the end of the proof of [Va11, Theorem 14.1] for a more clear exposition of this idea), we will consider an element $d \in \mathcal{U}(C)$ with sufficiently spread out Fourier coefficients so that for n large enough, $d(P_S \circ Q_{\mathcal{C}_n}(a_n))d^*$ is almost orthogonal to $P_S \circ Q_{\mathcal{C}_n}(a_n)$, while it is still close to $a_n = d a_n d^*$. Then the sum $d(P_S \circ Q_{\mathcal{C}_n}(a_n))d^* + P_S \circ Q_{\mathcal{C}_n}(a_n)$ should be even closer to a_n.

Let $\alpha > 0$ be a (finite) constant such that $\|x\|_\infty \leq \alpha \|x\|_2$, for all $x \in K_1$. Since $K_2 \subset A \ominus \mathbb{C}1$ is finite dimensional, the set

$$L = \{g \in \Gamma \mid \exists a, b \in K_2, \|a\|_2 \leq 1, \|b\|_2 \leq 1 : |\langle \sigma_g(a), b\rangle| \geq \varepsilon/|S|^2\lambda^2\alpha^2\}$$

is finite. Hence for all n, $L_n = \cup_{g,h\in F_n} gLh^{-1}$ is finite, with cardinality smaller or equal to $|F_n|^2|L|$, which is itself majorized by some N, not depending on n.

Since $C \not\prec A$, Ioana's intertwining criterion (Lemma A.3.6) implies that there exists $d \in \mathcal{U}(C)$ such that $\|P_F(d)\|_2 \leq \varepsilon/\kappa|S|$, whenever $|F| \leq N$, where κ is given by Lemma II.3.10 applied to K_1, K_2 and λ.

By Kaplansky's density theorem, one can find $d_0, d_1 \in M$, and $T \subset \Gamma$ finite such that:

- $d_i = P_T(d_i)$, $i = 1, 2$;
- $\|d_0 - d\|_2 \leq \min(\varepsilon, \varepsilon/\kappa|S|)$, $\|d_1 - d^*\|_2 \leq \varepsilon$;
- $\|d_i\|_\infty \leq 1$, $i = 1, 2$.

Since $a_n \in B$ for all n and $d \in C = B' \cap M$, we have $d a_n d^* = a_n$. Thus for all n, $\|a_n - d_0 a_n d_1\|_2 \leq 2\varepsilon$, and so

$$\limsup_n \|a_n - d_0(P_S \circ Q_{\mathcal{C}_n}(a_n))d_1\|_2 \leq \sqrt{\|p\|_2^2 - \delta^2} + 3\varepsilon. \tag{II.9}$$

Now, for all n, put $T_n = T \setminus L_n$. By definition of d and d_0, we have

$$\|d_0 - P_{T_n}(d_0)\|_2 \leq \|P_{L_n}(d_0)\|_2 \leq \|P_{L_n}(d)\|_2 + \varepsilon/\kappa|S| \leq 2\varepsilon/\kappa|S| \tag{II.10}$$

Notice that $\|P_S \circ Q_{\mathcal{C}_n}(a_n)\|_\infty \leq \kappa|S|$. Together with (II.9) and (II.10), this implies that

$$\limsup_n \|a_n - P_{T_n}(d_0)P_S \circ Q_{\mathcal{C}_n}(a_n)d_1\|_2 \leq \sqrt{\|p\|_2^2 - \delta^2} + 5\varepsilon.$$

Denote by $x_n = P_S \circ Q_{\mathcal{C}_n}(a_n)$ and $y_n = P_{T_n}(d_0)P_S \circ Q_{\mathcal{C}_n}(a_n)d_1$.

We want to show that $\limsup_n |\langle x_n, y_n\rangle|$ is small.

Write $d_0 = \sum_{g\in T} d_{0,g}u_g$, $a_n = \sum_h a_{n,h}u_h$, and $d_1 = \sum_{k\in T} d_{1,k}u_k$. We get

$$\begin{aligned}\langle y_n, x_n\rangle &= \sum_{\substack{g\in T_n, h\in S, k\in T\\ ghk\in S}} \tau(d_{0,g}\sigma_g(Q_{\mathcal{C}_n}(a_{n,h}))\sigma_{gh}(d_{1,k})Q_{\mathcal{C}_n}(a_{n,ghk})^*)\\ &= \sum_{\substack{g\in T, h\in S, k\in T\\ ghk\in S}} \mathbf{1}_{\{g\in T_n\}}\tau(d_{0,g}\sigma_{gh}(d_{1,k})\sigma_g(Q_{\mathcal{C}_n}(a_{n,h}))Q_{\mathcal{C}_n}(a_{n,ghk})^*).\end{aligned}$$

Claim. For all fixed $x, y \in A$, and $g \in T$, there exists n_0 such that for all $n \geq n_0$, and all $a, b \in \mathcal{C}_n$,

$$|\mathbf{1}_{\{g\in T_n\}}\tau(xy\sigma_g(a)b^*))| \leq 2\varepsilon\|x\|_2\|y\|_2/|S|^2.$$

To prove this claim, first recall that for all n, $\mathcal{C}_n = [K_1 \times \sigma_{F_n}(K_2)]^\lambda$. Denote by

$$\tilde{K}_1 = \mathrm{span}\{xy\sigma_g(a)b^*, a, b \in K_1\}.$$

Since $\tilde{K}_1$ and K_2 have finite dimension and since $F_n \to \infty$, Proposition II.3.4 implies that there exists n_0 such that for $n \geq n_0$, and for all $s, t \in F_n$ one has

$$\sup_{\substack{a\in\tilde{K}_1, \|a\|_2\leq 1\\ b,c\in K_2,\|b\|_2\leq 1,\|c\|_2\leq 1}} |\tau(a\sigma_{gs}(b)\sigma_t(c^*)) - \tau(a)\tau(\sigma_{gs}(b)\sigma_t(c^*))| \leq \varepsilon\|x\|_2\|y\|_2/|S|^2\lambda^2. \tag{II.11}$$

Thus take $n \geq n_0$. By definition of $\mathcal{C}_n$, it is sufficient to prove that for all $a, b \in K_1$, $c, d \in K_2$, with $\|a\|_2, \|b\|_2, \|c\|_2, \|d\|_2 \leq 1$, and all $s, t \in F_n$,

$$|\mathbf{1}_{\{g\in T_n\}}\tau(xy\sigma_g(\lambda a\sigma_s(c))\lambda b^*\sigma_t(d^*))| \leq 2\varepsilon\|x\|_2\|y\|_2/|S|^2.$$

We can assume that $g \in T_n$. An easy calculation gives

$$\begin{aligned}
|\tau(xy\sigma_g(\lambda a\sigma_s(c))\lambda b^*\sigma_t(d^*))| &\leq \varepsilon\|x\|_2\|y\|_2/|S|^2 + \lambda^2|\tau(xy\sigma_g(a)b^*)\tau(\sigma_{gs}(c)\sigma_t(d^*))| \\
&\leq \varepsilon\|x\|_2\|y\|_2/|S|^2 + \lambda^2\|x\|_2\|y\|_2\|a\|_\infty\|b\|_\infty\varepsilon/|S|^2\lambda^2\alpha^2 \\
&\leq 2\varepsilon\|x\|_2\|y\|_2/|S|^2,
\end{aligned}$$

where the first inequality is deduced from II.11, while the second is because $g \notin L_n$. So the claim is proved.

Now we can estimate $|\langle x_n, y_n\rangle|$, for n large enough.

$$\begin{aligned}
|\langle x_n, y_n\rangle| &\leq \sum_{g\in T, h\in S, k'\in S} |\mathbf{1}_{\{g\in T_n\}}\tau(d_{0,g}\sigma_{gh}(d_{1,h^{-1}g^{-1}k'})\sigma_g(Q_{C_n}(a_{n,h}))Q_{C_n}(a_{n,k'})^*)| \\
&\leq \sum_{g\in T, h\in S, k'\in S} 2\varepsilon\|d_{0,g}\|_2\|d_{1,h^{-1}g^{-1}k'}\|_2/|S|^2 \\
&\leq 2\varepsilon\|d_0\|_2\|d_1\|_2 \leq 2\varepsilon.
\end{aligned}$$

Therefore, we obtain:

- $\limsup_n \|a_n - x_n\|_2 < \sqrt{\|p\|_2^2 - \delta^2} + \varepsilon$;
- $\limsup_n \|a_n - y_n\|_2 < \sqrt{\|p\|_2^2 - \delta^2} + 5\varepsilon$;
- $\limsup_n |\langle x_n, y_n\rangle| \leq 2\varepsilon$.

Thus using the formula

$$\|x - (y+z)\|_2^2 = \|x-y\|_2^2 + \|x-z\|_2^2 - \|x\|_2^2 + 2\Re\langle y, z\rangle,$$

one checks that $\limsup_n \|a_n - (x_n + y_n)\|_2^2 \leq \|p\|_2^2 - 3\delta^2/2$, if ε is small enough. Let us show that $x_n + y_n$ belongs to some $\mathcal{C}'_n$ as in the conclusion of lemma.

Observe that

$$y_n = \sum_{g\in T_n, h\in S, k\in T} d_{0,g}\sigma_{gh}(d_{1,k})\sigma_g(Q_{\mathcal{C}_n}(a_{n,h}))u_{ghk}.$$

So let us check that y_n has its Fourier coefficients in the convex set $[K_0 \times \sigma_{TF_n}(K_2)]^{\lambda|S||T|}$, where $K_0 = \mathrm{span}\{d_{0,g}\sigma_{gh}(d_{1,k})\sigma_g(c), c \in K_1, g, k \in T, h \in S\}$ has finite dimension.

Fix $n \in \mathbb{N}$, and $s \in \Gamma$. Denote by $y_{n,s} = E_A(y_n u_s^*)$. We have

$$y_{n,s} = \sum_{\substack{g\in T_n, h\in S, k\in T \\ ghk=s}} d_{0,g}\sigma_{gh}(d_{1,k})\sigma_g(Q_{\mathcal{C}_n}(a_{n,h})).$$

Thus it is a convex combination of terms of the form

$$\begin{aligned}\mathcal{T} &= \sum_{\substack{g\in T, h\in S, k\in T \\ ghk=s}} d_{0,g}\sigma_{gh}(d_{1,k})\sigma_g(\lambda a_h \sigma_{t_h}(b_h)) \\ &= \frac{1}{|S||T|}\sum_{\substack{g\in T, h\in S, k\in T \\ ghk=s}} |S||T| d_{0,g}\sigma_{gh}(d_{1,k})\sigma_g(\lambda a_h \sigma_{t_h}(b_h)),\end{aligned}$$

for elements $a_h \in K_1$, $b_h \in K_2$, with $\|a_h\|_2, \|b_h\|_2 \leq 1$ and $t_h \in F_n$, for all $h \in S$. But such terms $\mathcal{T}$ are themselves convex combinations of elements of the form $\lambda|S||T|x\sigma_{gt}(y)$, with $x \in K_0, y \in K_2$, $\|x\|_2, \|y\|_2 \leq 1$ and $gt \in TF_n$.

Therefore, as pointed out in Lemma II.3.11, $x_n + y_n$ has Fourier coefficients in $\mathcal{C}'_n = [K'_1 \times \sigma_{F'_n}(K'_2)]^{\lambda'}$, with $K'_1 = K_1 + K_0$, $K'_2 = K_2$, $\lambda' = \lambda + \lambda|S||T|$, and $F'_n = F_n \cup TF_n$.

We conclude that:

$$\|a_n - Q_{C'_n}(a_n)\|_2^2 \leq \|p\|_2^2 - 3\delta^2/2,$$

which proves the lemma. □

The proof of Theorem II.3.1 is complete. □

The following question on a possible generalization of Theorem II.3.1 seems to be interesting.

Question II.3.13. *Consider a mixing Gaussian action* $\Gamma \curvearrowright (X,\mu)$ *of a discrete countable group* Γ. *Put* $A = L^\infty(X,\mu)$, *and* $M = A \rtimes \Gamma$. *Assume that* $B \subset M$ *is an abelian subalgebra which is normalized by a sequence of unitaries* $v_n \in \mathcal{U}(M)$ *such that:*

- *the deformation* $(\alpha_t)_t$ *converges uniformly on the set* $\{v_n\,,\, n \in \mathbb{N}\}$.
- (v_n) *goes weakly to* 0 *relative to* A: $\|E_A(xv_ny)\|_2 \to 0$ *for all* $x, y \in M$.

Is it true that either $B' \cap M \prec_M A$ *or* $B \prec_M L\Gamma$*?*

II.4 W*-rigidity

In this section we are interested in W*-rigidity results for Gaussian actions.

As explained in the introduction (Section I.2), W*-rigidity results are obtained by combining OE-rigidity results, and structural results for Cartan subalgebras in the associated crossed product.

II.4.1 Popa's OE-superrigidity results

Using the malleability property of Gaussian actions described above, Popa managed to prove the following striking OE-superrigidity theorems.

Theorem II.4.1 ([Po07a], Theorem 0.3)**.** *Assume that Γ is an ICC group which admits an infinite normal subgroup with the relative property (T).*

Then any mixing Gaussian action of Γ is OE-superrigid.

Theorem II.4.2 ([Po08], Corollary 1.3)**.** *Assume that $\Gamma = \Gamma_1 \times \Gamma_2$ is an ICC group, with Γ_1 non-amenable and Γ_2 infinite. Consider a mixing representation π such that $\pi^{\otimes k}$ is weakly contained in the regular representation for some $k \geq 1$.*

Then the Gaussian action of Γ associated with π is OE-superrigid.

II.4.2 First W*-rigidity results

With the work of Section II.3 we are able to generalize many W*-rigidity results about Bernoulli actions to general mixing Gaussian actions.

The first W*-rigidity result was [Po06b, Theorem 0.1]. We generalize this result as follows.

Theorem II.4.3. *Let Γ and Λ be two ICC countable discrete groups, and let $\pi : \Gamma \to \mathcal{O}(H)$ be a mixing orthogonal representation of Γ. Make one of the following two assumptions:*

(i) either Λ admits an infinite normal subgroup Λ_0 such that the pair (Λ, Λ_0) has the relative property (T);

(ii) or $\Lambda = \Lambda_1 \times \Lambda_2$, with Λ_1 non-amenable and Λ_2 infinite, and some tensor power of π is weakly contained in the regular representation.

Denote by $\Gamma \curvearrowright (X, \mu)$ the Gaussian action associated with π and consider a measure preserving action $\Lambda \curvearrowright (Y, \nu)$.

If there exists a $*$-isomorphism $L^\infty(Y, \nu) \rtimes \Lambda \simeq (L^\infty(X, \mu) \rtimes \Gamma)^t$ for some $0 < t \leq 1$, then $t = 1$ and the actions are conjugate.

Proof. Put $A = L^\infty(X, \mu)$, $B = L^\infty(Y, \nu)$, $M = A \rtimes \Gamma$ and $N = B \rtimes \Lambda$. By assumption, we have an identification $N = pMp$, with $p \in M$ a projection with trace t. We can assume that $p \in A$. We use the deformation (α_t) of M from Section II.2.1.

Step 1. There exists a unitary $u \in M$ such that $uL\Lambda u^* \subset L\Gamma$.

In case (i), we see that $L\Lambda_0$ is (α_t)-rigid inside pMp. In case (ii), spectral gap rigidity implies that $L\Lambda_2$ is (α_t)-rigid inside pMp.

In both cases, since Λ is non amenable, Corollary II.2.11 implies that there exists a unitary $u \in M$ such that $uL\Lambda u^* \subset L\Gamma$.

Step 2. The Cartan subalgebras B and pA in $N = pMp$ are unitarily conjugate.

Put $q = upu^*$. Note that uBu^* is normalized by the unitary elements $uv_g u^* \in N$, $g \in \Lambda$. Since $uL\Lambda u^* \subset L\Gamma$, a corner version of Theorem II.3.1 implies that either $(uBu^*)' \cap qMq \prec A$ or

$uBu^* \prec L\Gamma$. The latter case is impossible because Proposition A.4.6.1 would further imply that $pMp = N \prec L\Gamma$, which is clearly not true.

Therefore, we get $uBu^* = (uBu^*)' \cap qMq \prec A$ and then $B \prec_{pMp} pA$, which implies that these two Cartan subalgebras of pMp are conjugate by Theorem A.3.5.

STEP 3. Conclusion.

One concludes with Popa's conjugacy criterion [Po06b, Theorem 5.2]. $\square$

Remark II.4.4. In fact, the conclusion of our proof relies on Popa's conjugacy criterion which implies a more accurate result: any isomorphism $\phi : L^\infty(X,\mu) \rtimes \Gamma \to L^\infty(Y,\nu) \rtimes \Lambda$ *comes* from a conjugacy of the action in the sense that $\phi = \mathrm{Ad}(u) \circ \phi^\gamma \circ \phi^{\delta,\Delta}$, where $u \in \mathcal{U}(M)$ and:

- $\gamma : \Gamma \to \mathbb{C}$ is a character and $\phi^\gamma(au_g) = \gamma(g)au_g$ for all $a \in L^\infty(X,\mu)$, $g \in \Gamma$;
- $\delta : \Gamma \to \Lambda$ is a group isomorphism and $\Delta : (X,\mu) \to (Y,\nu)$ is a measure-preserving, bi-measurable isomorphism such that $\Delta(gx) = \delta(g)\Delta(x)$ for a.e. $x \in X$ and all $g \in \Gamma$. Finally, $\phi^{\delta,\Delta}(au_g) = (a \circ \Delta^{-1})v_{\delta(g)}$ for $a \in L^\infty(X,\mu)$, $g \in \Gamma$.

Theorem II.4.3 gives in particular a classification of all II_1-factors arising from mixing Gaussian actions of property (T) groups. It clearly implies that the fundamental group of such factors is trivial.

Being more accurate in the proof (see Remark II.4.4), we could also describe all $*$-endomorphisms of $A \rtimes \Gamma$ for Gaussian actions $\Gamma \curvearrowright A$ such that

- the initial representation is mixing and has some tensor power which is weakly contained in the regular representation;
- $\Gamma = \Gamma_1 \times \Gamma_2$, with Γ_1 non-amenable, Γ_2 is infinite, and Γ is ICC and has CMAP (or is weakly amenable).

The precise statement and its proof are similar to [Io11, Theorem 10.5], with the update [Oz12] allowing to replace the CMAP assumption by weak amenability.

II.4.3 W*-superrigidity for mixing Gaussian actions

Instead of classifying crossed product II_1-factors inside a specific class, a natural (but much harder) question is the W*-superrigidity question: is there an explicit class of crossed-product II_1-factors M such that an $*$-isomorphism of M with any other crossed-product von Neumann algebra implies conjugacy of the actions?

In other words, can one put all the assumptions "on the same side"?

The following theorem is our most general W*-rigidity result. It generalizes similar results for Bernoulli actions [Io11, Theorem A] and [IPV13, Theorem 10.1].

Theorem II.4.5. *Let Γ be an ICC countable discrete group, and let $\pi : \Gamma \to \mathcal{O}(H_\mathbb{R})$ be a mixing orthogonal representation of Γ. Make one of the following two assumptions:*

(i) Γ admits an infinite normal subgroup Γ_0 such that the pair (Γ, Γ_0) has the relative property (T);

(ii) *Γ is a non-amenable product of two infinite groups and π admits a tensor power which is weakly contained in the regular representation.*

Let $\Gamma \curvearrowright A$ be the Gaussian action associated with π and put $M = A \rtimes \Gamma$. Let $\Lambda \curvearrowright B$ be another free ergodic pmp action on an abelian von Neumann algebra, and put $N = B \rtimes \Lambda$.

If for some $t \geq 1$, $M \simeq N^t$, then $t = 1$, $\Gamma \simeq \Lambda$ and the actions $\Gamma \curvearrowright A$ and $\Lambda \curvearrowright B$ are conjugate.

In particular, $\Gamma \curvearrowright A$ is W-superrigid.*

Note that this implies that t-amplifications of M as in the theorem, $t > 1$, are never isomorphic to crossed-product von Neumann algebras. This feature was already observed by Ioana ([Io11]) in the case of Bernoulli actions. In the same vein, he proved that if Γ is as in case (i) and is torsion free, then no non-trivial amplification of $A \rtimes \Gamma$ (for the Bernoulli action $\Gamma \curvearrowright A$) is isomorphic to a (twisted) group von Neumann algebra. In the next section, we investigate the Gaussian case and we show that even M itself cannot be isomorphic to a group von Neumann algebra, for some Gaussian actions $\Gamma \curvearrowright A$.

With Theorem II.3.2 in hand, the proof of Theorem II.4.5 is very similar to what was done in the Bernoulli case, [IPV13, Theorem 10.1]. Since it is very technical, we only roughly explain the main steps of it; we refer to [Bo13, Section 4] and to the proof of [IPV13, Theorem 10.1] for technical details.

Steps of the proof of Theorem II.4.5. Let $\Gamma \curvearrowright X$ be a Gaussian action as Theorem II.4.5. Assume that $\Lambda \curvearrowright (Y, \nu)$ is another pmp, free ergodic action such that

$$L^\infty(X) \rtimes \Gamma \simeq L^\infty(Y) \rtimes \Lambda^2.$$

Put $A = L^\infty(X)$, $B = L^\infty(Y)$ and $M = A \rtimes \Gamma$.

Thanks to Popa's orbit equivalence superrigidity theorems (Theorem II.4.1 and Theorem II.4.2), we only need to show that the two actions are orbit equivalent. More concretely, with the results of Feldman and Moore [FM77] it is enough to prove that B is unitarily conjugate to A inside M.

The main idea of the proof, due to Ioana, is to exploit the information given by the isomorphism $M \simeq B \rtimes \Lambda$ via the dual co-action[3]

$$\begin{array}{rcl} \Delta : M & \to & M \mathbin{\overline{\otimes}} M \\ bv_s & \mapsto & bv_s \otimes v_s, \end{array}$$

$b \in B$, $s \in \Lambda$ (v_s, $s \in \Lambda$, denote the canonical unitaries corresponding to the action of Λ). This morphism Δ allows us to play against each other two data of the single action $\Gamma \curvearrowright X$: the rigidity of $\Delta(L\Gamma)$, and the malleability of the algebra $M \mathbin{\overline{\otimes}} M = (A \mathbin{\overline{\otimes}} A) \rtimes (\Gamma \times \Gamma)$.

Assume that B is not unitarily conjugate to A, or equivalently that $B \not\prec_M A$ by Theorem A.3.5. The following four steps lead to a contradiction, compare with the proof of Theorem II.4.3.

Step 1. There exists a unitary $u \in M \mathbin{\overline{\otimes}} M$ such that

$$u\Delta(L\Gamma)u^* \subset L\Gamma \mathbin{\overline{\otimes}} L\Gamma.$$

[2] We assume for simplicity that we are in the case $t = 1$.

[3] This morphism were also introduced by Popa and Vaes in [PV10a, Lemma 3.2].

This can be deduced from a tensor product version of Corollary II.2.11 exactly as in the proof of Step 1 of II.4.3.

STEP 2. The algebra $C := \Delta(A)' \cap (M \overline{\otimes} M)$ satisfies

$$C \prec_{M \overline{\otimes} M} A \overline{\otimes} A.$$

Indeed, the algebra $u\Delta(A)u^*$ is normalized by the unitaries $u\Delta(u_g)u^* \subset L\Gamma \overline{\otimes} L\Gamma$, $g \in \Gamma$. Applying Theorem II.3.2, it is not too hard to prove Step 2.

STEP 3. The previous steps, and an enhanced version of Popa's conjugacy criterion [Po06b, Theorem 5.2] (namely [IPV13, Theorem 6.1, Corollary 6.2]) roughly imply that there exists a unitary $v \in M \overline{\otimes} M$, a group homomorphism $\delta : \Gamma \to \Gamma \times \Gamma$, and a character $\omega : \Gamma \to \mathbb{C}$ such that

$$vCv^* = A \overline{\otimes} A \text{ and } v\Delta(u_g)v^* = \omega(g)u_{\delta(g)}, \forall g \in \Gamma.$$

STEP 4. Conclusion.

Using Step (3), one can now show that if a sequence (x_n) in M has Fourier coefficients (with respect to the decomposition $M = A \rtimes \Gamma$) which tend to zero pointwise in norm $\|\cdot\|_2$, then this is also the case of the sequence $\Delta(x_n)$, with respect to the decomposition $M \overline{\otimes} M = (M \overline{\otimes} A) \rtimes \Gamma$. This easily contradicts the fact that $B \not\prec_M A$. □

II.4.4 An application to group von Neumann algebras

We construct here a large class of II_1 factors which are not stably isomorphic to group von Neumann algebras.

Our examples are crossed-product von Neumann algebras of Gaussian actions associated with representations π as in Theorem II.4.5, with the extra-assumption that π is not weakly contained in the regular representation.

Whenever π is the regular representation, then the corresponding factor is of course a group factor, but Ioana, Popa and Vaes showed that all other amplifications are not isomorphic to group factors [IPV13, Theorem 8.2] (see also [Io11, Corollary 10.1]). Thanks to Theorem II.3.2, their result is also true for the representations π that we consider. So we only have to show that the factor itself is not a group factor.

Theorem II.4.6. *Let Γ and $\pi : \Gamma \to \mathcal{O}(H)$ be as in Theorem II.4.5. Assume moreover that π itself is not contained in a direct sum $\lambda^{\oplus\infty}$ of copies of the left-regular representation.*

Let $\Gamma \curvearrowright^{\sigma} A$ be the Gaussian action associated with π and put $M = A \rtimes \Gamma$. Then M is not stably isomorphic to a group von Neumann algebra.

Proof. Assume by contradiction that $M \simeq L\Lambda^t$ for some $t > 0$. Then adapting the proof of [IPV13, Theorem 8.2], we get that $t = 1$, and $\Lambda \simeq \Sigma \rtimes \Gamma$, for some infinite abelian group Σ and some action $\Gamma \curvearrowright \Sigma$ by automorphisms. Moreover, the initial Gaussian action σ is conjugate to the action of Γ on $L\Sigma$.

Now, since σ is mixing, the action $\Gamma \curvearrowright \Sigma \setminus \{e\}$ has finite stabilizers. But then the representation $\Gamma \curvearrowright \ell^2(\Sigma \setminus \{e\})$ is a direct sum of quasi-regular representations of the form $\Gamma \curvearrowright \ell^2(\Gamma/\Gamma_0)$, where Γ_0 is a finite subgroup of Γ. But such quasi-regular representations are all contained in the regular representation.

So we conclude that that the Koopman representation $\Gamma \curvearrowright L^2(A) \ominus \mathbb{C}1$ is contained in a direct sum of copies of the regular representation. Thus this is also the case of the sub-representation π, which is excluded by assumption. $\square$

By Proposition II.1.19, we know that for each $n \geq 3$, $\mathrm{SL}(n, \mathbb{Z})$ admits a representation as in Theorem C. Thus we obtain the existence of a II_1 factor M_n, which is not stably isomorphic to a group von Neumann algebra. But using Theorem II.4.3, we get that the M_n's are pairwise non-stably isomorphic : $M_n \ncong (M_m)^t$, $\forall t > 0$, $\forall n \neq m$.

Chapter III

Amalgamated free product type III factors with at most one Cartan subalgebra

This Chapter is based on a joint work with Cyril Houdayer and Sven Raum [BHR14]. We investigate Cartan subalgebras in nontracial amalgamated free product von Neumann algebras $M_1 *_B M_2$ over an amenable von Neumann subalgebra B. First, we settle the problem of the absence of Cartan subalgebra in arbitrary free product von Neumann algebras. Namely, we show that any nonamenable free product von Neumann algebra $(M_1, \varphi_1) * (M_2, \varphi_2)$ with respect to faithful normal states has no Cartan subalgebra. This generalizes the tracial case that was established in [Io(12)a]. Next, we prove that any countable nonsingular ergodic equivalence relation $\mathcal{R}$ defined on a standard measure space and which splits as the free product $\mathcal{R} = \mathcal{R}_1 * \mathcal{R}_2$ of recurrent subequivalence relations gives rise to a nonamenable factor $\mathrm{L}(\mathcal{R})$ with a unique Cartan subalgebra, up to unitary conjugacy. Finally, we prove unique Cartan decomposition for a class of group measure space factors $\mathrm{L}^\infty(X) \rtimes \Gamma$ arising from nonsingular free ergodic actions $\Gamma \curvearrowright (X, \mu)$ on standard measure spaces of amalgamated groups $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ over a finite subgroup Σ.

Contents

III.1 Introduction and main results

A *Cartan subalgebra* A in a von Neumann algebra M is a unital maximal abelian $*$-subalgebra $A \subset M$ such that there exists a faithful normal conditional expectation $E_A : M \to A$ and such that the group of normalizing unitaries of A inside M defined by $\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) : uAu^* = A\}$ generates M.

By a classical result of Feldman and Moore [FM77], any Cartan subalgebra A in a von Neumann algebra M with separable predual arises from a countable nonsingular equivalence relation $\mathcal{R}$ on a standard measure space (X, μ) and a 2-cocycle $v \in \mathrm{H}^2(\mathcal{R}, \mathbf{T})$. Namely, we have the following isomorphism of inclusions

$$(A \subset M) \cong (\mathrm{L}^\infty(X) \subset \mathrm{L}(\mathcal{R}, v)).$$

In particular, for any nonsingular free action $\Gamma \curvearrowright (X, \mu)$ of a countable discrete group Γ on a standard measure space (X, μ), $\mathrm{L}^\infty(X)$ is a Cartan subalgebra in the group measure space von Neumann algebra $\mathrm{L}^\infty(X) \rtimes \Gamma$.

The presence of a Cartan subalgebra A in a von Neumann algebra M with separable predual is therefore an important feature which allows to divide the classification problem for M up to $*$-isomorphism into two different questions: uniqueness of the Cartan subalgebra A inside M up to conjugacy and classification of the underlying countable nonsingular equivalence relation $\mathcal{R}$ up to orbit equivalence.

In [CFW81], Connes, Feldman and Weiss showed that any amenable countable nonsingular ergodic equivalence relation is hyperfinite and thus implemented by an ergodic $\mathbb{Z}$-action. This implies, together with [Kr76], that any two Cartan subalgebras inside an amenable factor are always conjugate by an automorphism.

The uniqueness of Cartan subalgebras up to conjugacy is no longer true in general for nonamenable factors. In [CJ82], Connes and Jones discovered the first examples of II_1 factors with at least two Cartan subalgebras which are not conjugate by an automorphism. More concrete examples were later found by Popa and Ozawa in [OP10b]. We also refer to the recent work of Speelman and Vaes [SV12] on II_1 factors with uncountably many non (stably) conjugate Cartan subalgebras.

In the last decade, Popa's *deformation/rigidity theory* [Po06a, Po06b, Po06c] has led to a lot of progress in the classification of II_1 factors arising from probability measure preserving (pmp) actions of countable discrete groups on standard probability spaces and from countable pmp equivalence relations. We refer to the recent surveys [Po07b, Va10a, Io(12)b] for an overview of this topic.

We highlight below three breakthrough results regarding uniqueness of Cartan subalgebras in nonamenable II_1 factors. In his pioneering article [Po06c], Popa showed that any *rigid* Cartan subalgebra inside group measure space II_1 factors $\mathrm{L}^\infty(X) \rtimes \mathbb{F}_n$ arising from *rigid* pmp free ergodic actions $\mathbb{F}_n \curvearrowright (X, \mu)$ of the free group $\mathbb{F}_n$ ($n \geq 2$) is necessarily unitarily conjugate to $\mathrm{L}^\infty(X)$. In [OP10a], Ozawa and Popa proved that any *compact* pmp free ergodic action of the free group $\mathbf{F}_n$ ($n \geq 2$) gives rise to a II_1 factor $\mathrm{L}^\infty(X) \rtimes \mathbb{F}_n$ with unique Cartan decomposition, up to unitary conjugacy. This was the first result in the literature proving the uniqueness of Cartan subalgebras in nonamenable II_1 factors. Recently, Popa and Vaes [PV(12)] proved that *any* pmp free ergodic action of the free group $\mathbb{F}_n$ ($n \geq 2$) gives rise to a II_1 factor $\mathrm{L}^\infty(X) \rtimes \mathbb{F}_n$

with unique Cartan decomposition, up to unitary conjugacy. We refer to [OP10b, Ho10, CS13, CSU13, PV(13), HV13, Io(12)a] for further results in this direction.

Very recently, using [PV(12)], Ioana [Io(12)a] obtained new results regarding the Cartan decomposition of *tracial* amalgamated free product von Neumann algebras $M_1 *_B M_2$. Let us highlight below two of Ioana's results [Io(12)a]: any nonamenable tracial free product $M_1 * M_2$ has no Cartan subalgebra and any pmp free ergodic action $\Gamma \curvearrowright (X, \mu)$ of a free product group $\Gamma = \Gamma_1 * \Gamma_2$ with $|\Gamma_1| \geq 2$ and $|\Gamma_2| \geq 3$ gives rise to a II_1 factor with unique Cartan decomposition, up to unitary conjugacy.

In the present paper, we use Popa's deformation/rigidity theory to investigate Cartan subalgebras in *nontracial* amalgamated free product (AFP) von Neumann algebras $M_1 *_B M_2$ over an amenable von Neumann subalgebra B. We generalize some of Ioana's recent results [Io(12)a] to this setting. The methods of proofs rely on a combination of results and techniques from [PV(12), HV13, Io(12)a].

Statement of the main results

Using his free probability theory, Voiculescu [Vo96] proved that the free group factors $\mathrm{L}(\mathbb{F}_n)$ $(n \geq 2)$ have no Cartan subalgebra. This exhibited the first examples of II_1 factors with no Cartan decomposition. This result was generalized later in [Ju07] to free product II_1 factors $M_1 * M_2$ of diffuse subalgebras which are embeddable into R^ω. Finally, the general case of arbitrary tracial free product von Neumann algebras was recently obtained in [Io(12)a] using Popa's deformation/rigidity theory.

The first examples of type III factors with no Cartan subalgebra were obtained in [Shl00] as a consequence of [Vo96]. Namely, it was shown that the unique free Araki-Woods factor of type III_λ $(0 < \lambda < 1)$ has no Cartan subalgebra. This result was vastly generalized later in [HR11] where it was proven that in fact any free Araki-Woods factor has no Cartan subalgebra.

Our first result settles the question of the absence of Cartan subalgebra in arbitrary free product von Neumann algebras.

Theorem III.A. *Let (M_1, φ_1) and (M_2, φ_2) be any von Neumann algebras with separable predual endowed with faithful normal states such that $\dim M_1 \geq 2$ and $\dim M_2 \geq 3$. Then the free product von Neumann algebra $(M, \varphi) = (M_1, \varphi_1) * (M_2, \varphi_2)$ has no Cartan subalgebra.*

Observe that when $\dim M_1 = \dim M_2 = 2$, the free product $M = M_1 * M_2$ is hyperfinite by [Dyk93, Theorem 1.1] and so has a Cartan subalgebra. Note that the questions of factoriality, type classification and fullness for arbitrary free product von Neumann algebras were recently settled in [Ue11]. These results are used in the proof of Theorem III.A.

We next investigate more generally Cartan subalgebras in nontracial AFP von Neumann algebras $M = M_1 *_B M_2$ over an amenable von Neumann subalgebra B. Even though we do not get a complete solution in that setting, our second result shows that, under fairly general assumptions, any Cartan subalgebra $A \subset M$ can be embedded into B inside M, in the sense of Popa's intertwining techniques. We refer to Section III.2 for more information on these intertwining techniques and the notation $A \preceq_M B$. Recall from [HV13, Definition 5.1] that an inclusion of von Neumann algebras $P \subset M$ has no *trivial corner* if for all nonzero projections $p \in P' \cap M$, we have $Pp \neq pMp$.

Theorem III.B. *For $i \in \{1, 2\}$, let $B \subset M_i$ be any inclusion of von Neumann algebras with separable predual and with faithful normal conditional expectation $E_i : M_i \to B$. Let $(M, E) = (M_1, E_1) *_B (M_2, E_2)$ be the corresponding amalgamated free product von Neumann algebra. Assume that B is a finite amenable von Neumann algebra.*

Assume moreover that:

- *Either both M_1 and M_2 have no amenable direct summand.*
- *Or B is of finite type* I*, M_1 has no amenable direct summand and the inclusion $B \subset M_2$ has no trivial corner.*

If $A \subset M$ is a Cartan subalgebra, then $A \preceq_M B$.

A similar result was obtained for tracial AFP von Neumann algebras in [Io(12)a, Theorem 1.3].

The first examples of type III factors with unique Cartan decomposition were recently obtained in [HV13]. Namely, it was shown that any nonamenable nonsingular free ergodic action $\Gamma \curvearrowright (X, \mu)$ of a Gromov hyperbolic group on a standard measure space gives rise to a factor $\mathrm{L}^\infty(X) \rtimes \Gamma$ with unique Cartan decomposition, up to unitary conjugacy. This generalized the probability measure preserving case that was established in [PV(13)].

In order to state our next results, we need to introduce some terminology. Let $\mathcal{R}$ be a countable nonsingular equivalence relation on a standard measure space (X, μ) and denote by $\mathrm{L}(\mathcal{R})$ the von Neumann algebra of the equivalence relation $\mathcal{R}$ ([FM77]). Following [Ad94, Definition 2.1], we say that $\mathcal{R}$ is *recurrent* if for all measurable subsets $\mathcal{U} \subset X$ such that $\mu(\mathcal{U}) > 0$, the set $[x]_\mathcal{R} \cap \mathcal{U}$ is infinite for almost every $x \in \mathcal{U}$. This is equivalent to saying that $\mathrm{L}(\mathcal{R})$ has no type I direct summand. We then say that a nonsingular action $\Gamma \curvearrowright (X, \mu)$ of a countable discrete group on a standard measure space is *recurrent* if the corresponding orbit equivalence relation $\mathcal{R}(\Gamma \curvearrowright X)$ is recurrent.

Our next result provides a new class of type III factors with unique Cartan decomposition, up to unitary conjugacy. These factors arise from countable nonsingular ergodic equivalence relations $\mathcal{R}$ which split as a free product $\mathcal{R} = \mathcal{R}_1 * \mathcal{R}_2$ of arbitrary recurrent subequivalence relations. We refer to [Ga00, Definition IV.6] for the notion of *free product* of countable nonsingular equivalence relations.

Theorem III.C. *Let $\mathcal{R}$ be any countable nonsingular ergodic equivalence relation on a standard measure space (X, μ) which splits as a free product $\mathcal{R} = \mathcal{R}_1 * \mathcal{R}_2$ such that the subequivalence relation $\mathcal{R}_i$ is recurrent for all $i \in \{1, 2\}$.*

Then the nonamenable factor $\mathrm{L}(\mathcal{R})$ has $\mathrm{L}^\infty(X)$ as its unique Cartan subalgebra, up to unitary conjugacy. In particular, for any nonsingular ergodic equivalence relation $\mathcal{S}$ on a standard measure space (Y, η) such that $\mathrm{L}(\mathcal{R}) \cong \mathrm{L}(\mathcal{S})$, we have $\mathcal{R} \cong \mathcal{S}$.

Observe that Theorem III.C generalizes [Io(12)a, Corollary 1.4] where the same result was obtained for countable pmp equivalence relations under additional assumptions. Note that in the case when $\mathcal{R}_1$ is *nowhere amenable*, that is, $\mathrm{L}(\mathcal{R}_1)$ has no amenable direct summand and $\mathcal{R}_2$ is recurrent, Theorem III.C is a consequence of Theorem III.B and [HV13, Theorem 2.5]. However, Theorem III.B does not cover the case when both $\mathcal{R}_1$ and $\mathcal{R}_2$ are amenable. So, in the setting of von Neumann algebras arising from countable nonsingular equivalence relations, Theorem III.C is a generalization of Theorem III.B in the sense that we are able to remove the nonamenability assumption on $M_1 = \mathrm{L}(\mathcal{R}_1)$.

Finally, when dealing with certain nonsingular free ergodic actions $\Gamma \curvearrowright (X, \mu)$ of amalgamated groups $\Gamma_1 *_\Sigma \Gamma_2$, we obtain new examples of group measure space type III factors with unique Cartan decomposition, up to unitary conjugacy.

Theorem III.D. *Let $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ be any amalgamated free product of countable discrete groups such that Σ is finite and Γ_i is infinite for all $i \in \{1, 2\}$. Let $\Gamma \curvearrowright (X, \mu)$ be any nonsingular free ergodic action on a standard measure space such that for all $i \in \{1, 2\}$, the restricted action $\Gamma_i \curvearrowright (X, \mu)$ is recurrent.*

Then the group measure space factor $\mathrm{L}^\infty(X) \rtimes \Gamma$ has $\mathrm{L}^\infty(X)$ as its unique Cartan subalgebra, up to unitary conjugacy.

Observe that Theorem III.D generalizes the probability measure preserving case that was established in [Io(12)a, Theorem 1.1].

In the spirit of [HV13, Corollary B], we obtain the following interesting consequence. Let $\Gamma = \Gamma_1 * \Gamma_2$ be an arbitrary free product group such that Γ_1 is amenable and infinite and $|\Gamma_2| \geq 2$. Then we get group measure space factors of the form $\mathrm{L}^\infty(X) \rtimes \Gamma$ with unique Cartan decomposition, having any possible type and with any possible flow of weights in the type III_0 case.

We finally mention that, unlike the probability measure preserving case [Io(12)a, Theorem 1.1], the assumption of recurrence of the action $\Gamma_i \curvearrowright (X, \mu)$ for all $i \in \{1, 2\}$ is necessary. Indeed, using [SV12], we exhibit in Section III.8 a class of nonamenable infinite measure preserving free ergodic actions $\Gamma \curvearrowright (X, \mu)$ of free product groups $\Gamma = \Gamma_1 * \Gamma_2$ such that the corresponding type II_∞ group measure space factor $\mathrm{L}^\infty(X) \rtimes \Gamma$ has uncountably many non conjugate Cartan subalgebras.

Comments on the proofs

As we already mentioned above, the proofs of our main results rely heavily on results and techniques from [PV(12), HV13, Io(12)a]. Let us describe below the main three ingredients which are needed. We will mainly focus on the proof of Theorem III.A.

Denote by $(M, \varphi) = (M_1, \varphi_1) * (M_2, \varphi_2)$ an arbitrary free product of von Neumann algebras as in Theorem III.A. For simplicity, we may assume that M is a factor. In the case when both M_1 and M_2 are amenable, M is already known to have no Cartan subalgebra by [HR11, Theorem 5.5]. So we may assume that M_1 is not amenable. Using [Dyk93, Ue11], we may further assume that M_1 has no amenable direct summand and $M_2 \neq \mathbb{C}$. By contradiction, assume that $A \subset M$ is a Cartan subalgebra.

We first use Connes-Takesaki's *noncommutative flow of weights* [Co73, CT77, Ta03] in order to work inside the semifinite von Neumann algebra $\mathrm{c}(M)$ which is the *continuous core* of M. We obtain a canonical decomposition of $\mathrm{c}(M)$ as the semifinite amalgamated free product von Neumann algebra $\mathrm{c}(M) = \mathrm{c}(M_1) *_{\mathrm{L}(\mathbb{R})} \mathrm{c}(M_2)$. Moreover $\mathrm{c}(A) \subset \mathrm{c}(M)$ is a Cartan subalgebra.

Next, we use Popa's intertwining techniques in the setting of nontracial von Neumann algebras that were developed in [HV13, Section 2]. Since A is diffuse, we show that necessarily $\mathrm{c}(A) \npreceq_{\mathrm{c}(M)} \mathrm{L}(\mathbb{R})$ (see Proposition III.2.10).

Finally, we extend Ioana's techniques from [Io(12)a, Sections 3,4] to *semifinite* AFP von Neumann algebras (see Theorems III.3.4 and III.4.1). The major difference though between our approach and Ioana's approach is that we cannot use the spectral gap techniques from [Io(12)a, Section 5]. The main reason why Ioana's approach cannot work here is that $\mathrm{c}(M)$ is not full in general even though M is a full factor. Instead, we strengthen [Io(12)a, Theorem 4.1] in the following way. We show that the presence of the Cartan subalgebra $\mathrm{c}(A) \subset \mathrm{c}(M)$ which satisfies $\mathrm{c}(A) \npreceq_{\mathrm{c}(M)} \mathrm{L}(\mathbb{R})$ forces *both* $\mathrm{c}(M_1)$ and $\mathrm{c}(M_2)$ to have an amenable direct summand. Therefore, both M_1 and M_2 have an amenable direct summand as well. Since we assumed that M_1 had no amenable direct summand, this is a contradiction.

III.2 Preliminaries

Since we want the paper to be as self contained as possible, we recall in this section all the necessary background that will be needed for the proofs of the main results.

III.2.1 Intertwining techniques

All the von Neumann algebras that we consider in this paper are always assumed to be σ-finite. Let M be a von Neumann algebra. We say that a von Neumann subalgebra $P \subset 1_P M 1_P$ is with *expectation* if there exists a faithful normal conditional expectation $E_P : 1_P M 1_P \to P$. Whenever $\mathcal{V} \subset M$ is a linear subspace, we denote by $\mathrm{Ball}(\mathcal{V})$ the *unit ball* of $\mathcal{V}$ with respect to the uniform norm $\|\cdot\|_\infty$. We will sometimes say that a von Neumann algebra (M, τ) is *tracial* if M is endowed with a faithful normal tracial state τ.

In [Po06a, Po06b, Po06c], Popa discovered the following powerful method to unitarily conjugate subalgebras of a finite von Neumann algebra. Let M be a finite von Neumann algebra and $A \subset 1_A M 1_A$, $B \subset 1_B M 1_B$ von Neumann subalgebras. By [Po06a, Corollary 2.3] and [Po06c, Theorem A.1], the following statements are equivalent:

- There exist projections $p \in A$ and $q \in B$, a nonzero partial isometry $v \in pMq$ and a unital normal $*$-homomorphism $\varphi : pAp \to qBq$ such that $av = v\varphi(a)$ for all $a \in A$.
- There exist $n \geq 1$, a possibly nonunital normal $*$-homomorphism $\pi : A \to \mathbf{M}_n(B)$ and a nonzero partial isometry $v \in \mathbf{M}_{1,n}(1_A M 1_B)$ such that $av = v\pi(a)$ for all $a \in A$.
- There is no net of unitaries (w_k) in $\mathcal{U}(A)$ such that $E_B(x^* w_k y) \to 0$ $*$-strongly for all $x, y \in 1_A M 1_B$.

If one of the previous equivalent conditions is satisfied, we say that A *embeds into* B *inside* M and write $A \preceq_M B$.

We will need the following generalization of Popa's Intertwining Theorem which was proven in [HV13, Theorems 2.3, 2.5]. A further generalization can also be found in [Ue(12), Proposition 3.1].

Theorem III.2.1. *Let M be any von Neumann algebra. Let $A \subset 1_A M 1_A$ and $B \subset 1_B M 1_B$ be von Neumann subalgebras such that B is finite and with expectation $E_B : 1_B M 1_B \to B$. The following are equivalent.*

1. *There exist $n \geq 1$, a possibly nonunital normal $*$-homomorphism $\pi : A \to \mathbf{M}_n(B)$ and a nonzero partial isometry $v \in \mathbf{M}_{1,n}(1_A M 1_B)$ such that $av = v\pi(a)$ for all $a \in A$.*

2. *There is no net of unitaries (w_k) in $\mathcal{U}(A)$ such that $E_B(x^* w_k y) \to 0$ $*$-strongly for all $x, y \in 1_A M 1_B$.*

Moreover, when M is a factor and $A, B \subset M$ are both Cartan subalgebras, the previous conditions are equivalent with the following:

(3) There exists a unitary $u \in \mathcal{U}(A)$ such that $uAu^* = B$.

Definition III.2.2. Let M be any von Neumann algebra. Let $A \subset 1_A M 1_A$ and $B \subset 1_B M 1_B$ be von Neumann subalgebras such that B is finite and with expectation. We say that A *embeds into* B *inside* M and denote $A \preceq_M B$ if one of the equivalent conditions of Theorem III.2.1 is satisfied.

Observe that when 1_A and 1_B are finite projections in M then $1_A \vee 1_B$ is finite, and $A \preceq_M B$ in the sense of Definition III.2.2 if and only if $A \preceq_{(1_A \vee 1_B) M (1_A \vee 1_B)} B$ holds in the usual sense for finite von Neumann algebras.

In case of semifinite von Neumann algebras, we recall the following useful intertwining result (see [HR11, Lemma 2.2]). When $(\mathcal{B}, \mathrm{Tr})$ is a semifinite von Neumann algebra endowed with a semifinite faithful normal trace, we will denote by $\mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$ the set of all nonzero finite trace projections of $\mathcal{B}$. We will denote by $\|\cdot\|_{2,\mathrm{Tr}}$ the L^2-norm associated with the trace Tr.

Lemma III.2.3. *Let $(\mathcal{M}, \mathrm{Tr})$ be a semifinite von Neumann algebra endowed with a semifinite faithful normal trace. Let $\mathcal{B} \subset \mathcal{M}$ be a von Neumann subalgebra such that $\mathrm{Tr}\,|\mathcal{B}$ is semifinite. Denote by $E_{\mathcal{B}} : \mathcal{M} \to \mathcal{B}$ the unique trace-preserving faithful normal conditional expectation.*

Let $p \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{M})$ and $\mathcal{A} \subset p\mathcal{M}p$ any von Neumann subalgebra. The following conditions are equivalent:

1. *For every $q \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$, we have $\mathcal{A} \npreceq_{\mathcal{M}} q\mathcal{B}q$.*
2. *There exists an increasing sequence of projections $q_n \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$ such that $q_n \to 1$ strongly and $\mathcal{A} \npreceq_{\mathcal{M}} q_n \mathcal{B} q_n$ for all $n \in \mathbb{N}$.*
3. *There exists a net of unitaries $w_k \in \mathcal{U}(\mathcal{A})$ such that $\lim_k \|E_{\mathcal{B}}(x^* w_k y)\|_{2,\mathrm{Tr}} = 0$ for all $x, y \in p\mathcal{M}$.*

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3) Let $\mathcal{F} \subset \mathrm{Ball}(p\mathcal{M})$ be a finite subset and $\varepsilon > 0$. We need to show that there exists $w \in \mathcal{U}(\mathcal{A})$ such that $\|E_{\mathcal{B}}(x^* w y)\|_{2,\mathrm{Tr}} < \varepsilon$ for all $x, y \in \mathcal{F}$. Since the projection p has finite trace, there exists $n \in \mathbb{N}$ large enough such that

$$\|q_n x^* p - x^* p\|_{2,\mathrm{Tr}} + \|p y q_n - p y\|_{2,\mathrm{Tr}} < \frac{\varepsilon}{2}, \forall x, y \in \mathcal{F}.$$

Put $q = q_n$. Since $\mathcal{A} \npreceq_{\mathcal{M}} q\mathcal{B}q$, there exists a net $w_k \in \mathcal{U}(\mathcal{A})$ such that $\lim_k \|E_{q\mathcal{B}q}(a^* w_k b)\|_{2,\mathrm{Tr}} = 0$ for all $a, b \in p\mathcal{M}q$. Appying this to $a = pxq$ and $b = pyq$, if we take $w = w_k$ for k large enough, we get $\|E_{\mathcal{B}}(qx^* p\, w\, pyq)\|_{2,\mathrm{Tr}} = \|E_{q\mathcal{B}q}(qx^* p\, w\, pyq)\|_{2,\mathrm{Tr}} < \frac{\varepsilon}{2}$. Therefore, $\|E_{\mathcal{B}}(x^* w y)\|_{2,\mathrm{Tr}} < \varepsilon$.

(3) $\Rightarrow$ (1) Let $q \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$ and put $e = p \vee q$. Let $\lambda = \mathrm{Tr}(e) < \infty$ and denote by $\|\cdot\|_2$ the L^2-norm with respect to the normalized trace on $e\mathcal{M}e$. For all $x, y \in p\mathcal{M}q$, we have

$$\lim_k \|E_{q\mathcal{B}q}(x^* w_k y)\|_2 = \lambda^{-1/2} \lim_k \|E_{q\mathcal{B}q}(x^* w_k y)\|_{2,\mathrm{Tr}} = 0.$$

This means exactly that $\mathcal{A} \npreceq_{e\mathcal{M}e} q\mathcal{B}q$ in the usual sense for tracial von Neumann algebras and so $\mathcal{A} \npreceq_{\mathcal{M}} q\mathcal{B}q$. ∎

Let Γ be any countable discrete group and $\mathcal{S}$ any nonempty collection of subgroups of Γ. Following [BO08, Definition 15.1.1], we say that a subset $\mathcal{F} \subset \Gamma$ is *small relative to* $\mathcal{S}$ if there exist $n \geq 1$, $\Sigma_1, \dots, \Sigma_n \in \mathcal{S}$ and $g_1, h_1, \dots, g_n, h_n \in \Gamma$ such that $\mathcal{F} \subset \bigcup_{i=1}^n g_i \Sigma_i h_i$.

We will need the following generalization of [Va13, Proposition 2.6] and [HV13, Lemma 2.7].

Proposition III.2.4. *Let $(\mathcal{B}, \mathrm{Tr})$ be a semifinite von Neumann algebra endowed with a semifinite faithful normal trace. Let $\Gamma \curvearrowright (\mathcal{B}, \mathrm{Tr})$ be a trace preserving action of a countable discrete group Γ on $(\mathcal{B}, \mathrm{Tr})$ and denote by $\mathcal{M} = \mathcal{B} \rtimes \Gamma$ the corresponding semifinite crossed product von Neumann algebra. Let $p \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{M})$ and $\mathcal{A} \subset p\mathcal{M}p$ any von Neumann subalgebra. Denote $\mathcal{P} = \mathcal{N}_{p\mathcal{M}p}(\mathcal{A})''$.*

For every subset $\mathcal{F} \subset \Gamma$ which is small relative to $\mathcal{S}$, denote by $P_{\mathcal{F}}$ the orhogonal projection from $\mathrm{L}^2(\mathcal{M}, \mathrm{Tr})$ onto the closed linear span of $\{x u_g : x \in \mathcal{B} \cap \mathrm{L}^2(\mathcal{B}, \mathrm{Tr}), g \in \mathcal{F}\}$.

1. *The set $\mathcal{J} = \{e \in \mathcal{A}' \cap p\mathcal{M}p : \mathcal{A}e \npreceq_{\mathcal{M}} q(\mathcal{B} \rtimes \Sigma)q, \forall \Sigma \in \mathcal{S}, \forall q \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})\}$ is directed and attains its maximum in a projection z which belongs to $\mathcal{Z}(\mathcal{P})$.*
2. *There exists a net (w_k) in $\mathcal{U}(\mathcal{A}z)$ such that $\lim_k \|P_{\mathcal{F}}(w_k)\|_{2,\mathrm{Tr}} = 0$ for every subset $\mathcal{F} \subset \Gamma$ which is small relative to $\mathcal{S}$.*
3. *For every $\varepsilon > 0$, there exists a subset $\mathcal{F} \subset \Gamma$ which is small relative to $\mathcal{S}$ such that $\|a - P_{\mathcal{F}}(a)\|_{2,\mathrm{Tr}} < \varepsilon$ for all $a \in \mathcal{A}(p - z)$.*

Proof. (1) In order to show that the set $\mathcal{J}$ is directed and attains its maximum, it suffices to prove that whenever $(e_i)_{i \in I}$ is a family of projections in $\mathcal{A}' \cap p\mathcal{M}p$ and $e = \bigvee_{i \in I} e_i$, if $e \notin \mathcal{J}$, then there exists $i \in I$ such that $e_i \notin \mathcal{J}$. If $e \notin \mathcal{J}$, there exist $\Sigma \in \mathcal{S}$ and $q \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$ such that $\mathcal{A}e \preceq_{\mathcal{M}} q(\mathcal{B} \rtimes \Sigma)q$. Let $n \geq 1$, a nonzero partial isometry $v \in \mathbf{M}_{1,n}(\mathbb{C}) \otimes e\mathcal{M}q$ and a normal $*$-homomorphism $\varphi : \mathcal{A}e \to \mathbf{M}_n(q(\mathcal{B} \rtimes \Sigma)q)$ such that $av = v\varphi(a)$ for all $a \in \mathcal{A}e$. By definition we have $ev = v$. Choose $i \in I$ such that $e_i v \neq 0$ and denote by $w \in \mathbf{M}_{1,n}(\mathbb{C}) \otimes e_i\mathcal{M}q$ the polar part of $e_i v$. Since $aw = w\varphi(a)$ for all $a \in \mathcal{A}e$, it follows that $\mathcal{A}e_i \preceq_{\mathcal{M}} q(\mathcal{B} \rtimes \Sigma)q$. Hence, $e_i \notin \mathcal{J}$.

Denote by z the maximum of the set $\mathcal{J}$. It is easy to see that $uzu^* \in \mathcal{J}$ whenever $u \in \mathcal{N}_{p\mathcal{M}p}(\mathcal{A})$, hence $uzu^* = z$. Therefore $z \in \mathcal{Z}(\mathcal{P})$.

(2) We have that $\mathcal{A}z \npreceq_{\mathcal{M}} q(\mathcal{B} \rtimes \Sigma)q$ for all $\Sigma \in \mathcal{S}$ and all $q \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$. Let $\varepsilon > 0$ and $\mathcal{F} \subset \Gamma$ a subset which is small relative to $\mathcal{S}$. We show that we can find $w \in \mathcal{U}(\mathcal{A}z)$ such that $\|P_{\mathcal{F}}(w)\|_{2,\mathrm{Tr}} < \varepsilon$.

Let $\mathcal{F} \subset \bigcup_{i=1}^n g_i \Sigma_i h_i$ with $\Sigma_1, \dots, \Sigma_n \in \mathcal{S}$ and $g_1, h_1, \dots, g_n, h_n \in \Gamma$. Consider the semifinite von Neumann algebra $\mathbf{M}_n(\mathcal{M})$ together with the diagonal subalgebra $\mathcal{Q} = \bigoplus_{i=1}^n \mathcal{B} \rtimes \Sigma_i$. Observe that the canonical trace on $\mathbf{M}_n(\mathcal{M})$ is still semifinite on $\mathcal{Q}$. Consider moreover the trace preserving $*$-embedding $\rho : \mathcal{M} \to \mathbf{M}_n(\mathcal{M}) : x \mapsto x \oplus \cdots \oplus x$.

Since $\mathcal{A}z \npreceq_{\mathcal{M}} q(\mathcal{B} \rtimes \Sigma_i)q$ for all $i \in \{1, \dots, n\}$ and all $q \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$, we get that $\rho(\mathcal{A}z) \npreceq_{\mathbf{M}_n(\mathcal{M})} \rho(q)\mathcal{Q}\rho(q)$ for all $q \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$ by the first criterion in Lemma III.2.3. Then by the second criterion in Lemma III.2.3, there exists a net $w_k \in \mathcal{U}(\mathcal{A}z)$ such that

$$\lim_k \|E_{\mathcal{B} \rtimes \Sigma_i}(x w_k y)\|_{2,\mathrm{Tr}} = 0, \forall x, y \in \mathcal{M}, \forall i \in \{1, \dots, n\}.$$

Recall that $P_{g\Sigma h}(x) = u_g E_{\mathcal{B} \rtimes \Sigma}(u_g^* x u_h^*) u_h$ for all $x \in \mathcal{M} \cap \mathrm{L}^2(\mathcal{M}, \mathrm{Tr})$. Applying what we have just proved to $x = u_{g_i}^*$ and $y = u_{h_i}^*$, we get that $\lim_k \|P_{g_i \Sigma_i h_i}(w_k)\|_{2,\mathrm{Tr}} = 0$ for all $i \in \{1, \dots, n\}$. Therefore $\lim_k \|P_{\mathcal{F}}(w_k)\|_{2,\mathrm{Tr}} = 0$.

(3) By construction, for any projection $e \leq p - z$, there exist $\Sigma \in \mathcal{S}$ and $q \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$ such that $\mathcal{A}e \preceq_{\mathcal{M}} q(\mathcal{B} \rtimes \Sigma)q$. Let $\varepsilon > 0$. Choose $\ell \geq 1$ and $e_1, \dots, e_\ell \in \mathcal{A}' \cap p\mathcal{M}p$ pairwise orthogonal projections such that:

- For every $i \in \{1, \dots, \ell\}$, $e_i \leq p - z$ and $e = e_1 + \cdots + e_\ell$ satisfies $\|(p-z) - e\|_{2,\mathrm{Tr}} \leq \varepsilon/3$.
- For every $i \in \{1, \dots, \ell\}$, there exist $n_i \geq 1$, $\Sigma_i \in \mathcal{S}$, a projection $q_i \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$, a nonzero partial isometry $v_i \in \mathbf{M}_{1,n_i}(\mathbb{C}) \otimes e_i\mathcal{M}q_i$ and a normal $*$-homomorphism $\varphi_i : \mathcal{A} \to \mathbf{M}_{n_i}(q_i(\mathcal{B} \rtimes \Sigma_i)q_i)$ such that $v_i v_i^* = e_i$ and $av_i = v_i\varphi_i(a)$ for all $a \in \mathcal{A}$.

Put $n = n_1 + \cdots + n_\ell$, $q = \bigvee_{i=1}^{\ell} q_i$ and define $\varphi : \mathcal{A} \to \bigoplus_{i=1}^{\ell} q_i(\mathcal{B} \rtimes \Sigma_i)q_i \subset \mathbf{M}_n(q\mathcal{M}q)$ by putting together the φ_i diagonally. Similarly, define the partial isometry $v \in \mathbf{M}_{1,n}(\mathbb{C}) \otimes e\mathcal{M}q$ such that $vv^* = e$ and $av = v\varphi(a)$ for all $a \in \mathcal{A}$.

Using Kaplansky density theorem, choose $v_0 \in \mathbf{M}_{1,n}(\mathbb{C}) \otimes q(\mathcal{B} \rtimes_{\mathrm{alg}} \Gamma)q$ such that $\|v_0\|_\infty \leq 1$ and $\|v - v_0\|_{2,\mathrm{Tr}} < \varepsilon/3$. Define $\mathcal{G} \subset \Gamma$ the finite subset such that v_0 belongs to the linear span of $\{e_{1i} \otimes exu_g q : x \in \mathcal{B}, g \in \mathcal{G}, 1 \leq i \leq \ell\}$. Put $\mathcal{F} = \bigcup_{i=1}^{\ell} \bigcup_{g,h \in \mathcal{G}} g\Sigma_i h^{-1}$.

Let $a \in \mathrm{Ball}(\mathcal{A}(p-z))$ and write $a = a(p-z-e) + ae$. Observe that $\|a(p-z-e)\|_{2,\mathrm{Tr}} \leq \|a\|_\infty \|p-z-e\|_{2,\mathrm{Tr}} < \varepsilon/3$. Since $ae = v\varphi(a)v^*$, it follows that ae lies at a distance less than $2\varepsilon/3$ from $v_0\varphi(a)v_0^*$. Observe that by construction $P_{\mathcal{F}}(v_0\varphi(a)v_0^*) = v_0\varphi(a)v_0^*$. Therefore, a lies at a distance less than ε from the range of $P_{\mathcal{F}}$. $\square$

III.2.2 Amalgamated free product von Neumann algebras

For $i \in \{1, 2\}$, let $B \subset M_i$ be an inclusion of von Neumann algebras with expectation $E_i : M_i \to B$. Recall that the *amalgamated free product* $(M, E) = (M_1, E_1) *_B (M_2, E_2)$ is the von Neumann algebra M generated by M_1 and M_2 where the faithful normal conditional expectation $E : M \to B$ satisfies the freeness condition:

$$E(x_1 \cdots x_n) = 0 \text{ whenever } x_j \in M_{i_j} \ominus B \text{ and } i_j \neq i_{j+1} \, .$$

Here and in what follows, we denote by $M_i \ominus B$ the kernel of the conditional expectation $E_i : M_i \to B$. We refer to [Vo85, Vo92, Ue99] for more details on the construction of amalgamated free products in the framework of von Neumann algebras.

Assume that Tr is a semifinite faithful normal trace on B such that for all $i \in \{1, 2\}$, the weight $\mathrm{Tr} \circ E_i$ is a trace on M_i. Then the weight $\mathrm{Tr} \circ E$ is a trace on M by [Ue99, Theorem 2.6]. In that case, we will say that the amalgamated free product $(M, E) = (M_1, E_1) *_B (M_2, E_2)$ is *semifinite*. Whenever we consider a semifinite faithful normal trace on a semifinite amalgamated free product $(M, E) = (M_1, E_1) *_B (M_2, E_2)$, we will always assume that $\mathrm{Tr} \circ E = \mathrm{Tr}$ and $\mathrm{Tr}\,|B$ is semifinite.

The following proposition is a semifinite analogue of [IPP08, Theorem 1.1]. The proof of Theorem III.2.5 is essentially contained in [CH10, Theorem 2.4].

Theorem III.2.5. *Let $(\mathcal{M}, E) = (\mathcal{M}_1, E_1) *_{\mathcal{B}} (\mathcal{M}_2, E_2)$ be a semifinite amalgamated free product von Neumann algebra with semifinite faithful normal trace* Tr. *Let $p \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{M}_1)$ and $\mathcal{Q} \subset p\mathcal{M}_1 p$ any von Neumann subalgebra. Assume that there exists a net of unitaries $w_k \in \mathcal{U}(\mathcal{Q})$ such that $\lim_k \|E_{\mathcal{B}}(x^* w_k y)\|_{2,\mathrm{Tr}} = 0$ for all $x, y \in p\mathcal{M}_1$.*

Then any $\mathcal{Q}$-$p\mathcal{M}_1 p$-subbimodule $\mathcal{H}$ of $\mathrm{L}^2(p\mathcal{M}p)$ which has finite dimension as a right $p\mathcal{M}_1 p$-bimodule must be contained in $\mathrm{L}^2(p\mathcal{M}_1 p)$. In particular, $\mathcal{N}_{p\mathcal{M}p}(\mathcal{Q})'' \subset p\mathcal{M}_1 p$.

Proof. Using [Ta02, Proposition V.2.36], we denote by $E_{\mathcal{M}_1} : \mathcal{M} \to \mathcal{M}_1$ the unique trace preserving faithful normal conditional expectation which satisfies

$$E_{\mathcal{M}_1}(x_1 \cdots x_{2m+1}) = 0$$

whenever $m \geq 1$, $x_1, x_{2m+1} \in \mathcal{M}_1$, $x_{2j} \in \mathcal{M}_2 \ominus \mathcal{B}$ and $x_{2j+1} \in \mathcal{M}_1 \ominus \mathcal{B}$ for all $1 \leq j \leq m-1$. Observe that we moreover have $\mathrm{Tr} \circ E_{\mathcal{M}_1} = \mathrm{Tr}$. We denote by $\mathcal{M} \ominus \mathcal{M}_1$ the kernel of the conditional expectation $E_{\mathcal{M}_1} : \mathcal{M} \to \mathcal{M}_1$.

Claim. We have that $\lim_k \|E_{\mathcal{M}_1}(x^* w_k y)\|_{2,\mathrm{Tr}} = 0$ for all $x, y \in p(\mathcal{M} \ominus \mathcal{M}_1)$.

Proof of the Claim. Observe that using Kaplansky's density theorem, it suffices to prove the Claim for $x = px_1 \cdots x_{2m+1}$ and $y = py_1 \cdots y_{2n+1}$ with $m, n \geq 1$, $x_1, x_{2m+1}, y_1, y_{2n+1} \in \mathcal{M}_1$, $x_{2\ell+1}, y_{2\ell'+1} \in \mathcal{M}_1 \ominus \mathcal{B}$ and $x_{2\ell}, y_{2\ell'} \in \mathcal{M}_2 \ominus \mathcal{B}$ for all $1 \leq \ell \leq m-1$ and all $1 \leq \ell' \leq n-1$. Then, we have

$$E_{\mathcal{M}_1}(x^* w_k y) = E_{\mathcal{M}_1}(x_{2m+1}^* \cdots x_2^* \, E_{\mathcal{B}}(x_1^* w_k y_1) \, y_2 \cdots y_{2n+1}).$$

Hence, $\lim_k \|E_{\mathcal{M}_1}(x^* w_k y)\|_{2,\mathrm{Tr}} = 0$. □

In particular, we get $\lim_k \|E_{p\mathcal{M}_1 p}(x^* w_k y)\|_{2,\mathrm{Tr}} = 0$ for all $x, y \in p\mathcal{M}p \ominus p\mathcal{M}_1 p$. Finally, applying [Va07, Lemma D.3], we are done. □

We will moreover need the following technical results.

Proposition III.2.6. *Let $(\mathcal{M}, E) = (\mathcal{M}_1, E_1) *_{\mathcal{B}} (\mathcal{M}_2, E_2)$ be a semifinite amalgamated free product von Neumann algebra with semifinite faithful normal trace* Tr. *Assume the following:*

- *For all $i \in \{1, 2\}$ and all nonzero projections $z \in \mathcal{Z}(\mathcal{B})$, $\mathcal{B}z \neq z\mathcal{M}_i z$.*
- *For all $p \in \mathrm{Proj_f}(\mathcal{M})$ and all $q \in \mathrm{Proj_f}(\mathcal{B})$, we have $p\mathcal{M}p \not\preceq_{\mathcal{M}} q\mathcal{B}q$.*

Then for all $i \in \{1, 2\}$, all $e \in \mathrm{Proj_f}(\mathcal{M})$ and all $f \in \mathrm{Proj_f}(\mathcal{M}_i)$, we have $e\mathcal{M}e \not\preceq_{\mathcal{M}} f\mathcal{M}_i f$.

Proof. By contradiction, assume that there exist $i \in \{1, 2\}$, $e \in \mathrm{Proj_f}(\mathcal{M})$ and $f \in \mathrm{Proj_f}(\mathcal{M}_i)$, a nonzero partial isometry $v \in e\mathcal{M}f$ and a unital normal $*$-homomorphism $\varphi : e\mathcal{M}e \to f\mathcal{M}_i f$ such that $xv = v\varphi(x)$ for all $x \in e\mathcal{M}e$. We may assume without loss of generality that $i = 1$. Moreover, as in [Va08, Remark 3.8], we may assume that the support projection of $E_{\mathcal{M}_1}(v^* v)$ in $\mathcal{M}_1$ equals f.

Let $q \in \mathrm{Proj_f}(\mathcal{B})$ be arbitrary. By assumption, we have $e\mathcal{M}e \not\preceq_{\mathcal{M}} q\mathcal{B}q$. Next, we claim that $\varphi(e\mathcal{M}e) \not\preceq_{\mathcal{M}_1} q\mathcal{B}q$. Indeed, otherwise there would exist $n \geq 1$, a nonzero partial isometry $w \in \mathbf{M}_{1,n}(\mathbb{C}) \otimes f\mathcal{M}_1 q$ and a normal $*$-homomorphism $\psi : \varphi(e\mathcal{M}e) \to \mathbf{M}_n(q\mathcal{B}q)$ such that $\varphi(x)w = w\psi(\varphi(x))$ for all $x \in e\mathcal{M}e$. Hence, we get $xvw = vw(\psi \circ \varphi)(x)$ for all $x \in e\mathcal{M}e$. We have $E_{\mathbf{M}_n(\mathcal{M}_1)}(w^* v^* v w) = w^* E_{\mathcal{M}_1}(v^* v) w \neq 0$ since the support projection of $E_{\mathcal{M}_1}(v^* v)$ is f and $fw = w$. By taking the polar part of vw, this would imply that $e\mathcal{M}e \preceq_{\mathcal{M}} q\mathcal{B}q$, a contradiction.

By Lemma III.2.3 and Theorem III.2.5, we get $\varphi(e\mathcal{M}e)' \cap f\mathcal{M}f \subset f\mathcal{M}_1 f$, hence $v^* v \in f\mathcal{M}_1 f$. Thus, we may assume that $v^* v = f$. We get $f\mathcal{M}f = v^* \mathcal{M} v \subset f\mathcal{M}_1 f \subset f\mathcal{M}f$, so $f\mathcal{M}_1 f = f\mathcal{M}f$. The proof of [HV13, Theorem 5.7] shows that there exists a nonzero projection $z \in \mathcal{Z}(\mathcal{B})$ such that $z\mathcal{M}_2 z = \mathcal{B}z$, contradicting the assumptions. □

Proposition III.2.7. *Let $(\mathcal{M}, E) = (\mathcal{M}_1, E_1) *_{\mathcal{B}} (\mathcal{M}_2, E_2)$ be a semifinite amalgamated free product von Neumann algebra with semifinite faithful normal trace* Tr. *Let $p \in \mathrm{Proj_f}(\mathcal{M})$ and $\mathcal{A} \subset p\mathcal{M}p$ any von Neumann subalgebra. Assume there exist $i \in \{1, 2\}$ and $p_i \in \mathrm{Proj_f}(\mathcal{M}_i)$ such that $\mathcal{A} \preceq_{\mathcal{M}} p_i\mathcal{M}_i p_i$.*

Then either there exists $q \in \mathrm{Proj_f}(\mathcal{B})$ such that $\mathcal{A} \preceq_{\mathcal{M}} q\mathcal{B}q$ or $\mathcal{N}_{p\mathcal{M}p}(\mathcal{A})'' \preceq_{\mathcal{M}} p_i\mathcal{M}_i p_i$.

Proof. We assume that for all $q \in \mathrm{Proj_f}(\mathcal{B})$, we have $\mathcal{A} \not\preceq_{\mathcal{M}} q\mathcal{B}q$ and show that necessarily $\mathcal{N}_{p\mathcal{M}p}(\mathcal{A})'' \preceq_{\mathcal{M}} p_i\mathcal{M}_i p_i$.

Since $\mathcal{A} \preceq_{\mathcal{M}} p_i\mathcal{M}_i p_i$, there exist $n \geq 1$, a nonzero partial isometry $v \in \mathbf{M}_{1,n}(\mathbb{C}) \otimes p\mathcal{M}p_i$ and a possibly nonunital normal $*$-homomorphism $\varphi : \mathcal{A} \to \mathbf{M}_n(p_i\mathcal{M}_i p_i)$ such that $av = v\varphi(a)$ for all $a \in \mathcal{A}$. Since we also have $\mathcal{A} \not\preceq_{\mathcal{M}} q\mathcal{B}q$ for all $q \in \mathrm{Proj_f}(\mathcal{B})$, a reasoning entirely analogous to the one of the proof of Proposition III.2.6 allows us to further assume that $\varphi(\mathcal{A}) \not\preceq_{\mathbf{M}_n(\mathcal{M}_i)} \mathbf{M}_n(q\mathcal{B}q)$ for all $q \in \mathrm{Proj_f}(\mathcal{B})$.

Let $u \in \mathcal{N}_{p\mathcal{M}p}(\mathcal{A})$. Then for all $a \in \mathcal{A}$, we have

$$v^*uv\varphi(a) = vuav = v^*(uau^*)uv = \varphi(uau^*)v^*uv.$$

By Theorem III.2.5 and Lemma III.2.3, we get $v^*uv \in \mathbf{M}_n(p_i\mathcal{M}_i p_i)$ for all $u \in \mathcal{N}_{p\mathcal{M}p}(\mathcal{A})$, hence $v^*\mathcal{N}_{p\mathcal{M}p}(\mathcal{A})''v \subset p_i\mathcal{M}_i p_i$. Therefore, we have $\mathcal{N}_{p\mathcal{M}p}(\mathcal{A})'' \preceq_{\mathcal{M}} p_i\mathcal{M}_i p_i$. □

III.2.3 Hilbert bimodules

Let M and N be any von Neumann algebras. Recall that an M-N-*bimodule* $\mathcal{H}$ is a Hilbert space endowed with two commuting normal $*$-representations $\pi : M \to \mathbf{B}(\mathcal{H})$ and $\rho : N^{\mathrm{op}} \to \mathbf{B}(\mathcal{H})$. We then define $\pi_{\mathcal{H}} : M \otimes_{\mathrm{alg}} N^{\mathrm{op}} \to \mathbf{B}(\mathcal{H})$ by $\pi_{\mathcal{H}}(x \otimes y^{\mathrm{op}}) = \pi(x)\rho(y^{\mathrm{op}})$ for all $x \in M$ and all $y \in N$. We will simply write $x\xi y = \pi_{\mathcal{H}}(x \otimes y^{\mathrm{op}})\xi$ for all $x \in M$, all $y \in N$ and all $\xi \in \mathcal{H}$.

Let $\mathcal{H}$ and $\mathcal{K}$ be M-N-bimodules. Following [Co94, Appendix V.B], we say that $\mathcal{K}$ is *weakly contained* in $\mathcal{H}$ and write $\mathcal{K} \subset_{\mathrm{weak}} \mathcal{H}$ if $\|\pi_{\mathcal{K}}(T)\|_\infty \leq \|\pi_{\mathcal{H}}(T)\|_\infty$ for all $T \in M \otimes_{\mathrm{alg}} N^{\mathrm{op}}$. We simply denote by $(N, \mathrm{L}^2(N), J, \mathfrak{P})$ the standard form of N (see e.g. [Ta03, Chapter IX.1]). Then the N-N-bimodule $\mathrm{L}^2(N)$ with left and right action given by $x\xi y = xJy^*J\xi$ is the *trivial* N-N-bimodule while the N-N-bimodule $\mathrm{L}^2(N) \otimes \mathrm{L}^2(N)$ with left and right action given by $x(\xi \otimes \eta)y = x\xi \otimes Jy^*J\eta$ is the *coarse* N-N-bimodule.

Recall that a von Neumann algebra N is *amenable* if as N-N-bimodules, we have $\mathrm{L}^2(N) \subset_{\mathrm{weak}} \mathrm{L}^2(N) \otimes \mathrm{L}^2(N)$. Equivalently, there exists a norm one projection $\Phi : \mathbf{B}(\mathrm{L}^2(N)) \to N$.

For any von Neumann algebras B, M, N, any M-B-bimodule $\mathcal{H}$ and any B-N-bimodule $\mathcal{K}$, there is a well defined M-N-bimodule $\mathcal{H} \otimes_B \mathcal{K}$ called the Connes's fusion tensor product of $\mathcal{H}$ and $\mathcal{K}$ over B. We refer to [Co94, Appendix V.B] and [AD95, Section 1] for more details regarding this construction.

We will be using the following well known fact (see [AD95, Lemma 1.7]). For any von Neumann algebras B, M, N such that B is amenable, any M-B-bimodule $\mathcal{H}$ and any B-N-bimodule $\mathcal{K}$, we have, as M-N-bimodules,

$$\mathcal{H} \otimes_B \mathcal{K} \subset_{\mathrm{weak}} \mathcal{H} \otimes \mathcal{K}.$$

III.2.4 Relative amenability

Let M be any von Neumann algebra. Denote by $(M, \mathrm{L}^2(M), J, \mathfrak{P})$ the standard form of M. Let $P \subset 1_P M 1_P$ (resp. $Q \subset M$) be a von Neumann subalgebra with expectation $E_P : 1_P M 1_P \to P$ (resp. $E_Q : M \to Q$). The *basic construction* $\langle M, Q\rangle$ is the von Neumann algebra $(JQJ)' \cap \mathbf{B}(H)$. Following [OP10a, Section 2.1], we say that P is *amenable relative to* Q *inside* M if there exists a norm one projection $\Phi : 1_P\langle M, Q\rangle 1_P \to P$ such that $\Phi | 1_P M 1_P = E_P$.

In the case when (M, τ) is a tracial von Neumann algebra and the conditional expectation $E_P : M \to P$ (resp. $E_Q : M \to Q$) is τ-preserving, the basic construction that we denote by $\langle M, e_Q\rangle$ coincides with the von Neumann algebra generated by M and the orthogonal projection $e_Q : \mathrm{L}^2(M, \tau) \to \mathrm{L}^2(Q, \tau|Q)$. Observe that $\langle M, e_Q\rangle$ comes with a semifinite faithful normal trace given by $\mathrm{Tr}(x e_Q y) = \tau(xy)$ for all $x, y \in M$. Then [OP10a, Theorem 2.1] shows that P is amenable relative to Q inside M if and only if there exists a net of vectors $\xi_n \in \mathrm{L}^2(\langle M, e_Q\rangle, \mathrm{Tr})$ such that $\lim_n \langle x\xi_n, \xi_n\rangle_{\mathrm{Tr}} = \tau(x)$ for all $x \in 1_P M 1_P$ and $\lim_n \|y\xi_n - \xi_n y\|_{2,\mathrm{Tr}} = 0$ for all $y \in P$.

III.2.5 Noncommutative flow of weights

Let (M, φ) be a von Neumann algebra together with a faithful normal state. Denote by M^φ the centralizer of φ and by $M \rtimes_\varphi \mathbb{R}$ the *continuous core* of M, that is, the crossed product of M with the modular automorphism group $(\sigma_t^\varphi)_{t\in\mathbb{R}}$ associated with the faithful normal state φ. We have a canonical $*$-embedding $\pi_\varphi : M \to M \rtimes_\varphi \mathbb{R}$ and a canonical group of unitaries $(\lambda_\varphi(s))_{s\in\mathbb{R}}$ in $M \rtimes_\varphi \mathbb{R}$ such that

$$\pi_\varphi(\sigma_s^\varphi(x)) = \lambda_\varphi(s)\, \pi_\varphi(x)\, \lambda_\varphi(s)^* \quad \text{for all } \ x \in M, s \in \mathbb{R}.$$

The unitaries $(\lambda_\varphi(s))_{s\in\mathbb{R}}$ generate a copy of $\mathrm{L}(\mathbb{R})$ inside $M \rtimes_\varphi \mathbb{R}$.

We denote by $\widehat{\varphi}$ the *dual weight* on $M \rtimes_\varphi \mathbb{R}$ (see [Ta03, Definition X.1.16]), which is a semifinite faithful normal weight on $M \rtimes_\varphi \mathbb{R}$ whose modular automorphism group $(\sigma_t^{\widehat{\varphi}})_{t\in\mathbb{R}}$ satisfies

$$\sigma_t^{\widehat{\varphi}}(\pi_\varphi(x)) = \pi_\varphi(\sigma_t^\varphi(x)) \text{ for all } x \in M \quad \text{and} \quad \sigma_t^{\widehat{\varphi}}(\lambda_\varphi(s)) = \lambda_\varphi(s) \text{ for all } s \in \mathbb{R}.$$

We denote by $(\theta_t^\varphi)_{t\in\mathbb{R}}$ the *dual action* on $M \rtimes_\varphi \mathbb{R}$, given by

$$\theta_t^\varphi(\pi_\varphi(x)) = \pi_\varphi(x) \text{ for all } x \in M \quad \text{and} \quad \theta_t^\varphi(\lambda_\varphi(s)) = \exp(\mathrm{i}ts)\lambda_\varphi(s) \text{ for all } s \in \mathbb{R}.$$

Denote by h_φ the unique nonsingular positive selfadjoint operator affiliated with $\mathrm{L}(\mathbb{R}) \subset M \rtimes_\varphi \mathbb{R}$ such that $h_\varphi^{\mathrm{i}s} = \lambda_\varphi(s)$ for all $s \in \mathbb{R}$. Then $\mathrm{Tr}_\varphi = \widehat{\varphi}(h_\varphi^{-1} \cdot)$ is a semifinite faithful normal trace on $M \rtimes_\varphi \mathbb{R}$ and the dual action θ^φ *scales* the trace Tr_φ:

$$\mathrm{Tr}_\varphi \circ \theta_t^\varphi = \exp(t)\, \mathrm{Tr}_\varphi, \forall t \in \mathbb{R}.$$

Note that Tr_φ is semifinite on $\mathrm{L}(\mathbb{R}) \subset M \rtimes_\varphi \mathbb{R}$. Moreover, the canonical faithful normal conditional expectation $E_{\mathrm{L}(\mathbb{R})} : M \rtimes_\varphi \mathbb{R} \to \mathrm{L}(\mathbb{R})$ defined by $E_{\mathrm{L}(\mathbb{R})}(x\lambda_\varphi(s)) = \varphi(x)\lambda_\varphi(s)$ preserves the trace Tr_φ, that is,

$$\mathrm{Tr}_\varphi \circ E_{\mathrm{L}(\mathbb{R})} = \mathrm{Tr}_\varphi .$$

Because of Connes's Radon-Nikodym cocycle theorem (see [Ta03, Theorem VIII.3.3]), the semifinite von Neumann algebra $M \rtimes_\varphi \mathbb{R}$, together with its trace Tr_φ and trace-scaling action θ^φ, "does not depend" on the choice of φ in the following precise sense. If ψ is another faithful normal state on M, there is a canonical surjective $*$-isomorphism $\Pi_{\psi,\varphi} : M \rtimes_\varphi \mathbb{R} \to M \rtimes_\psi \mathbb{R}$ such that $\Pi_{\psi,\varphi} \circ \pi_\varphi = \pi_\psi$, $\mathrm{Tr}_\psi \circ \Pi_{\psi,\varphi} = \mathrm{Tr}_\varphi$ and $\Pi_{\psi,\varphi} \circ \theta^\varphi = \theta^\psi \circ \Pi_{\psi,\varphi}$. Note however that $\Pi_{\psi,\varphi}$ does not map the subalgebra $\mathrm{L}(\mathbb{R}) \subset M \rtimes_\varphi \mathbb{R}$ onto the subalgebra $\mathrm{L}(\mathbb{R}) \subset M \rtimes_\psi \mathbb{R}$.

Altogether we can abstractly consider the *continuous core* $(\mathrm{c}(M), \theta, \mathrm{Tr})$, where $\mathrm{c}(M)$ is a von Neumann algebra with a faithful normal semifinite trace Tr, θ is a trace-scaling action of $\mathbb{R}$ on $(\mathrm{c}(M), \mathrm{Tr})$ and $\mathrm{c}(M)$ contains a copy of M. Whenever φ is a faithful normal state on M, we get a canonical surjective $*$-isomorphism $\Pi_\varphi : M \rtimes_\varphi \mathbb{R} \to \mathrm{c}(M)$ such that

$$\Pi_\varphi \circ \theta^\varphi = \theta \circ \Pi_\varphi, \quad \mathrm{Tr}_\varphi = \mathrm{Tr} \circ \Pi_\varphi, \quad \Pi_\varphi(\pi_\varphi(x)) = x \ \ \forall x \in M.$$

A more functorial construction of the continuous core, known as the *noncommutative flow of weights* can be given, see [Co73, CT77, FT01].

By Takesaki's duality theorem [Ta03, Theorem X.2.3], we have that $\mathrm{c}(M) \rtimes_\theta \mathbb{R} \cong M \mathbin{\overline{\otimes}} \mathbf{B}(\mathrm{L}^2(\mathbb{R}))$. In particular, by [AD95, Proposition 3.4], M is amenable if and only if $\mathrm{c}(M)$ is amenable.

If $P \subset 1_P M 1_P$ is a von Neumann subalgebra with expectation, we have a canonical trace preserving inclusion $\mathrm{c}(P) \subset 1_P \mathrm{c}(M) 1_P$.

We will also frequently use the following well-known fact: if $A \subset M$ is a Cartan subalgebra then $\mathrm{c}(A) \subset \mathrm{c}(M)$ is still a Cartan subalgebra. For a proof of this fact, see e.g. [HR11, Proposition 2.6].

Proposition III.2.8. *Let M be any von Neumann algebra with no amenable direct summand. Then the continuous core $\mathrm{c}(M)$ has no amenable direct summand either.*

Proof. Assume that $\mathrm{c}(M)$ has an amenable direct summand. Let $z \in \mathcal{Z}(\mathrm{c}(M))$ be a nonzero projection such that $\mathrm{c}(M)z$ is amenable. Denote by $\theta : \mathbb{R} \curvearrowright \mathrm{c}(M)$ the dual action which scales the trace Tr. Put $e = \bigvee_{t \in \mathbb{R}} \theta_t(z)$. Observe that $e \in \mathcal{Z}(\mathrm{c}(M))$ and $\theta_t(e) = e$ for all $t \in \mathbb{R}$. By [Ta03, Theorem XII.6.10], we have $e \in M \cap \mathcal{Z}(\mathrm{c}(M))$, hence $e \in \mathcal{Z}(M)$. We canonically have $\mathrm{c}(M)e = \mathrm{c}(Me)$.

Since amenability is stable under direct limits, we have that $\mathrm{c}(M)e$ is amenable, hence $\mathrm{c}(Me)$ is amenable. Applying again [Ta03, Theorem XII.6.10], we have $\mathrm{c}(Me) \rtimes_\theta \mathbb{R} \cong (Me) \mathbin{\overline{\otimes}} \mathbf{B}(\mathrm{L}^2(\mathbb{R}))$. We get that $\mathrm{c}(Me) \rtimes_\theta \mathbb{R}$ is amenable and so is Me. Therefore, M has an amenable direct summand. $\square$

We will frequently use the following:

Notation III.2.9. Let $A \subset M$ (resp. $B \subset M$) be a von Neumann subalgebra with expectation $E_A : M \to A$ (resp. $E_B : M \to B$) of a given von Neumann algebra M. Assume moreover that A and B are both tracial. Let τ_A be a faithful normal trace on A (resp. τ_B on B) and write $\varphi_A = \tau_A \circ E_A$ (resp. $\varphi_B = \tau_B \circ E_B$). Write $\pi_{\varphi_A} : M \to M \rtimes_{\varphi_A} \mathbb{R}$ (resp. $\pi_{\varphi_B} : M \to M \rtimes_{\varphi_B} \mathbb{R}$) for the canonical $*$-representation of M into its continuous core associated with φ_A (resp. φ_B).

By Connes's Radon-Nikodym cocycle theorem, there is a surjective $*$-isomorphism

$$\Pi_{\varphi_B, \varphi_A} : M \rtimes_{\varphi_A} \mathbb{R} \to M \rtimes_{\varphi_B} \mathbb{R}$$

which intertwines the dual actions, that is, $\theta^{\varphi_B} \circ \Pi_{\varphi_B,\varphi_A} = \Pi_{\varphi_B,\varphi_A} \circ \theta^{\varphi_A}$, and preserves the faithful normal semifinite traces, that is, $\mathrm{Tr}_{\varphi_B} \circ \Pi_{\varphi_B,\varphi_A} = \mathrm{Tr}_{\varphi_A}$. In particular, we have $\Pi_{\varphi_B,\varphi_A}(\pi_{\varphi_A}(x)) = \pi_{\varphi_B}(x)$ for all $x \in M$.

Put $\mathrm{c}(M) = M \rtimes_{\varphi_B} \mathbb{R}$, $\mathrm{c}(B) = B \rtimes_{\varphi_B} \mathbb{R}$ and $\mathrm{c}(A) = \Pi_{\varphi_B,\varphi_A}(A \rtimes_{\varphi_A} \mathbb{R})$. We simply denote by $\mathrm{Tr} = \mathrm{Tr}_{\varphi_B}$ the canonical semifinite faithful normal trace on $\mathrm{c}(M)$. Observe that Tr is still semifinite on $\mathcal{Z}(\mathrm{c}(A))$ and $\mathcal{Z}(\mathrm{c}(B))$.

Proposition III.2.10. *Assume that we are in the setup of Notation III.2.9. If $A \npreceq_M B$, then for all $p \in \mathrm{Proj_f}(\mathcal{Z}(\mathrm{c}(A)))$ and all $q \in \mathrm{Proj_f}(\mathcal{Z}(\mathrm{c}(B)))$, we have $\mathrm{c}(A)p \npreceq_{\mathrm{c}(M)} \mathrm{c}(B)q$.*

Proof. Let $v_k \in \mathcal{U}(A)$ be a net such that $E_B(x^* v_k y) \to 0$ $*$-strongly for all $x, y \in M$. Recall that $\mathrm{c}(M) = M \rtimes_{\varphi_B} \mathbb{R}$, $\mathrm{c}(B) = B \rtimes_{\varphi_B} \mathbb{R}$ and $\mathrm{c}(A) = \Pi_{\varphi_B,\varphi_A}(A \rtimes_{\varphi_A} \mathbb{R})$. Let $p \in \mathrm{Proj_f}(\mathcal{Z}(\mathrm{c}(A)))$ and $q \in \mathrm{Proj_f}(\mathcal{Z}(\mathrm{c}(B)))$. Observe that since p commutes with every element in $\mathrm{c}(A)$, p commutes with every element in $\Pi_{\varphi_B,\varphi_A}(\pi_{\varphi_A}(A)) = \pi_{\varphi_B}(A) \subset \mathrm{c}(A)$. Then $w_k = \Pi_{\varphi_B,\varphi_A}(\pi_{\varphi_A}(v_k))p = \pi_{\varphi_B}(v_k)p$ is a net of unitaries in $\mathcal{U}(\mathrm{c}(A)p)$.

Write $\mathrm{c}(M)_{\mathrm{alg}} = M \rtimes^{\mathrm{alg}}_{\varphi_B} \mathbb{R}$ for the algebraic crossed product, that is, the linear span of $\{\pi_{\varphi_B}(x)\lambda_{\varphi_B}(t) : x \in M, t \in \mathbb{R}\}$. Observe that $\mathrm{c}(M)_{\mathrm{alg}}$ is a dense unital $*$-subalgebra of $\mathrm{c}(M)$. We have $E_{\mathrm{c}(B)}(x^*\pi_{\varphi_B}(v_k)y) \to 0$ $*$-strongly for all $x, y \in \mathrm{c}(M)_{\mathrm{alg}}$. Since $q \in \mathrm{Proj_f}(\mathrm{c}(B))$, we have

$$\|E_{\mathrm{c}(B)q}(q\, x^*\pi_{\varphi_B}(v_k) y\, q)\|_{2,\mathrm{Tr}} = \|q E_{\mathrm{c}(B)}(x^*\pi_{\varphi_B}(v_k)y)q\|_{2,\mathrm{Tr}} \to 0, \forall x, y \in \mathrm{c}(M)_{\mathrm{alg}}.$$

Fix now $x, y \in \mathrm{Ball}(\mathrm{c}(M))$. By Kaplansky density theorem, choose a net $(x_i)_{i \in I}$ (resp. $(y_j)_{j \in J}$) in $\mathrm{Ball}(\mathrm{c}(M)_{\mathrm{alg}})$ such that $x_i \to px$ (resp. $y_j \to py$) $*$-strongly. Let $\varepsilon > 0$. Since $q \in \mathrm{Proj_f}(\mathrm{c}(B))$, we can choose $(i, j) \in I \times J$ such that

$$\|(py - y_j)q\|_{2,\mathrm{Tr}} + \|q(x^*p - x_j)\|_{2,\mathrm{Tr}} < \varepsilon.$$

Therefore, by triangle inequality, we obtain

$$\limsup_k \|E_{\mathrm{c}(B)q}(q\, x^*\, p\pi_{\varphi_B}(v_k)p\, y\, q)\|_{2,\mathrm{Tr}} \leq \limsup_k \|E_{\mathrm{c}(B)q}(q\, x_i^*\pi_{\varphi_B}(v_k)y_j\, q)\|_{2,\mathrm{Tr}} + \varepsilon \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get $\lim_k \|E_{\mathrm{c}(B)q}(qx^*p\, w_k\, pyq)\|_{2,\mathrm{Tr}} = 0$. This finally proves that $\mathrm{c}(A)p \npreceq_{\mathrm{c}(M)} \mathrm{c}(B)q$. ☐

Example III.2.11. We emphasize two well-known examples that will be extensively used in this paper.

1. Let $\Gamma \curvearrowright (X, \mu)$ be any nonsingular action on a standard measure space. Define the *Maharam extension* (see [Ma64]) $\Gamma \curvearrowright (X \times \mathbb{R}, m)$ by

$$g \cdot (x, t) = \left(gx, t + \log\left(\frac{\mathrm{d}(\mu \circ g^{-1})}{\mathrm{d}\mu}(x)\right)\right),$$

   where $\mathrm{d}m = \mathrm{d}\mu \times \exp(t)\mathrm{d}t$. It is easy to see that the action $\Gamma \curvearrowright X \times \mathbb{R}$ preserves the infinite measure m and we moreover have that

$$\mathrm{c}(\mathrm{L}^\infty(X) \rtimes \Gamma) = \mathrm{L}^\infty(X \times \mathbb{R}) \rtimes \Gamma.$$

2. Let $(M, E) = (M_1, E_1) *_B (M_2, E_2)$ be any amalgamated free product von Neumann algebra. Fix a faithful normal state φ on B. We still denote by φ the faithful normal state $\varphi \circ E$ on M. We realize the continuous core of M as $\mathrm{c}(M) = M \rtimes_\varphi \mathbb{R}$. Likewise, if we denote by $\varphi_i = \varphi \circ E_i$ the corresponding state on M_i, we realize the continuous core of M_i

as $\mathrm{c}(M_i) = M_i \rtimes_{\varphi_i} \mathbb{R}$. We denote by $\mathrm{c}(E) : \mathrm{c}(M) \to \mathrm{c}(B)$ (resp. $\mathrm{c}(E_i) : \mathrm{c}(M_i) \to \mathrm{c}(B)$) the canonical trace preserving faithful normal conditional expectation. Recall from [Ue99, Section 2] that $\sigma_t^{\varphi}(M_i) = M_i$ for all $t \in \mathbb{R}$ and all $i \in \{1, 2\}$, hence

$$(\mathrm{c}(M), \mathrm{c}(E)) = (\mathrm{c}(M_1), \mathrm{c}(E_1)) *_{\mathrm{c}(B)} (\mathrm{c}(M_2), \mathrm{c}(E_2)).$$

Moreover, $\mathrm{c}(M)$ is a semifinite amalgamated free product von Neumann algebra.

III.3 Intertwining subalgebras inside semifinite AFP von Neumann algebras

III.3.1 Malleable deformation on semifinite AFP von Neumann algebras

First, we recall the construction of the malleable deformation on amalgamated free product von Neumann algebras discovered in [IPP08, Section 2].

Let $(\mathcal{M}, E) = (\mathcal{M}_1, E_1) *_{\mathcal{B}} (\mathcal{M}_2, E_2)$ be any semifinite amalgamated free product von Neumann algebra with semifinite faithful normal trace Tr. We will simply write $\mathcal{M} = \mathcal{M}_1 *_{\mathcal{B}} \mathcal{M}_2$ when no confusion is possible. Put $\widetilde{\mathcal{M}} = \mathcal{M} *_{\mathcal{B}} (\mathcal{B} \mathbin{\overline{\otimes}} \mathrm{L}(\mathbb{F}_2))$ and observe that $\widetilde{\mathcal{M}}$ is still a semifinite amalgamated free product von Neumann algebra. We still denote by Tr the semifinite faithful normal trace on $\widetilde{\mathcal{M}}$. Let $u_1, u_2 \in \mathcal{U}(\mathrm{L}(\mathbf{F}_2))$ be the canonical Haar unitaries generating $\mathrm{L}(\mathbb{F}_2)$. Observe that we can decompose $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}_1 *_{\mathcal{B}} \widetilde{\mathcal{M}}_2$ with $\widetilde{\mathcal{M}}_i = \mathcal{M}_i *_{\mathcal{B}} (\mathcal{B} \mathbin{\overline{\otimes}} \mathrm{L}(\mathbb{Z}))$.

Consider the unique Borel function $f : \mathbf{T} \to (-\pi, \pi]$ such that $f(\exp(\mathrm{i}t)) = t$ for all $t \in (-\pi, \pi]$. Define the selfadjoint operators $h_1 = f(u_1)$ and $h_2 = f(u_2)$ so that $\exp(\mathrm{i}u_1) = h_1$ and $\exp(\mathrm{i}u_2) = h_2$. For every $t \in \mathbb{R}$, put $u_1^t = \exp(\mathrm{i}th_1)$ and $u_2^t = \exp(\mathrm{i}th_2)$. We have

$$\tau(u_1^t) = \tau(u_2^t) = \frac{\sin(\pi t)}{\pi t}, \forall t \in \mathbb{R}.$$

Define the one-parameter group of trace preserving $*$-automorphisms $\alpha_t \in \mathrm{Aut}(\widetilde{\mathcal{M}})$ by

$$\alpha_t = \mathrm{Ad}(u_1^t) *_{\mathcal{B}} \mathrm{Ad}(u_2^t), \forall t \in \mathbb{R}.$$

Define moreover the trace preserving $*$-automorphism $\beta \in \mathrm{Aut}(\widetilde{\mathcal{M}})$ by

$$\beta = \mathrm{id}_{\mathcal{M}} *_{\mathcal{B}} (\mathrm{id}_{\mathcal{B}} \mathbin{\overline{\otimes}} \beta_0)$$

with $\beta_0(u_1) = u_1^*$ and $\beta_0(u_2) = u_2^*$. We have $\alpha_t \beta = \beta \alpha_{-t}$ for all $t \in \mathbb{R}$. Thus, (α_t, β) is a malleable deformation in the sense of Popa [Po07b].

We will be using the following notation throughout this section.

Notation III.3.1. Put $\mathcal{H}_0 = \mathrm{L}^2(\mathcal{B}, \mathrm{Tr})$ and $\mathcal{K}_0 = \mathrm{L}^2(\mathcal{B} \mathbin{\overline{\otimes}} \mathrm{L}(\mathbb{F}_2), \mathrm{Tr})$. For $n \geq 1$, define $S_n = \{(i_1, \dots, i_n) : i_1 \neq \cdots \neq i_n\}$ to be the set of the two alternating sequences of length n made of 1's and 2's. For $\mathcal{I} = (i_1, \dots, i_n) \in S_n$, denote by

- $\mathcal{H}_{\mathcal{I}}$ the closed linear span in $\mathrm{L}^2(\mathcal{M}, \mathrm{Tr})$ of elements $x_1 \cdots x_n$, with $x_j \in \mathcal{M}_{i_j} \ominus \mathcal{B}$ such that $\mathrm{Tr}(x_j^* x_j) < \infty$ for all $j \in \{1, \dots, n\}$.
- $\mathcal{K}_{\mathcal{I}}$ the closed linear span in $\mathrm{L}^2(\widetilde{\mathcal{M}}, \mathrm{Tr})$ of elements $u_{h_1} x_1 \cdots u_{h_n} x_n u_{h_{n+1}}$, with $h_j \in \mathbb{F}_2$ for all $j \in \{1, \dots, n+1\}$ and $x_j \in \mathcal{M}_{i_j} \ominus \mathcal{B}$ such that $\mathrm{Tr}(x_j^* x_j) < \infty$ for all $j \in \{1, \dots, n\}$.

We denote by $E_{\mathcal{M}} : \widetilde{\mathcal{M}} \to \mathcal{M}$ the unique trace preserving faithful normal conditional expectation as well as the orthogonal projection $\mathrm{L}^2(\widetilde{\mathcal{M}}, \mathrm{Tr}) \to \mathrm{L}^2(\mathcal{M}, \mathrm{Tr})$. We still denote by $\alpha : \mathbb{R} \to \mathcal{U}(\mathrm{L}^2(\widetilde{\mathcal{M}}, \mathrm{Tr}))$ the Koopman representation associated with the trace preserving action $\alpha : \mathbb{R} \to \mathrm{Aut}(\widetilde{\mathcal{M}})$.

Lemma III.3.2. *Let $m, n \geq 1$, $\mathcal{I} = (i_1, \dots, i_m) \in S_m$ and $\mathcal{J} = (j_1, \dots, j_n) \in S_n$. Let $x_1 \in \mathcal{M}_{i_1} \ominus \mathcal{B}, \dots, x_m \in \mathcal{M}_{i_m} \ominus \mathcal{B}$ and $y_1 \in \mathcal{M}_{j_1} \ominus \mathcal{B}, \dots, y_n \in \mathcal{M}_{j_n} \ominus \mathcal{B}$ with $\mathrm{Tr}(a^*a) < \infty$ for $a = x_1, \dots, x_m, y_1, \dots, y_n$. Let $g_1, \dots, g_{m+1}, h_1, \dots, h_{n+1} \in \mathbb{F}_2$. Then*

$$\langle u_{g_1} x_1 \cdots u_{g_m} x_m u_{g_{m+1}}, u_{h_1} y_1 \cdots u_{h_n} y_n u_{h_{n+1}} \rangle_{\mathrm{L}^2(\widetilde{\mathcal{M}}, \mathrm{Tr})} =$$

$$\begin{cases} \langle x_1 \cdots x_m, y_1 \cdots y_n \rangle_{\mathrm{L}^2(\mathcal{M}, \mathrm{Tr})} & \text{if } m = n,\ \mathcal{I} = \mathcal{J} \text{ and } g_k = h_k, \forall k \in \{1, \dots, m+1\}; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The proof is the same as the proof of [Io(12)a, Lemma 3.1]. We leave it to the reader. ☐

Lemma III.3.2 allows us, in particular, to put $\mathcal{H}_n = \bigoplus_{\mathcal{I} \in S_n} \mathcal{H}_{\mathcal{I}}$ and $\mathcal{K}_n = \bigoplus_{\mathcal{I} \in S_n} \mathcal{K}_{\mathcal{I}}$ since the $\mathcal{K}_{\mathcal{I}}$'s are pairwise orthogonal. We then have

$$\mathrm{L}^2(\mathcal{M}, \mathrm{Tr}) = \bigoplus_{n \in \mathbb{N}} \mathcal{H}_n \quad \text{and} \quad \mathrm{L}^2(\widetilde{\mathcal{M}}, \mathrm{Tr}) = \bigoplus_{n \in \mathbb{N}} \mathcal{K}_n.$$

For all $\xi \in \mathrm{L}^2(\mathcal{M}, \mathrm{Tr})$, write $\xi = \sum_{n \in \mathbb{N}} \xi_n$ with $\xi_n \in \mathcal{H}_n$ for all $n \in \mathbb{N}$. A simple calculation shows that for all $t \in \mathbb{R}$,

$$\mathrm{Tr}(\alpha_t(\xi)\xi^*) = \mathrm{Tr}(E_{\mathcal{M}}(\alpha_t(\xi))\xi^*) = \sum_{n \in \mathbb{N}} \left(\frac{\sin(\pi t)}{\pi t} \right)^{2n} \|\xi_n\|_{2,\mathrm{Tr}}^2.$$

Observe that $t \mapsto \mathrm{Tr}(\alpha_t(\xi)\xi^*)$ is decreasing on $[0, 1]$ for all $\xi \in \mathrm{L}^2(\mathcal{M}, \mathrm{Tr})$.

III.3.2 A semifinite analogue of Ioana-Peterson-Popa's intertwining theorem [IPP08]

The first result of this section is an analogue of the main technical result of [IPP08] (see [IPP08, Theorem 4.3]) for semifinite amalgamated free product von Neumann algebras. A similar result also appeared in [CH10, Theorem 4.2]. For the sake of completeness, we will give the proof.

Theorem III.3.3. *Let $\mathcal{M} = \mathcal{M}_1 *_{\mathcal{B}} \mathcal{M}_2$ be a semifinite amalgamated free product von Neumann algebra with semifinite faithful normal trace* Tr. *Let $p \in \mathrm{Proj_f}(\mathcal{M})$ and $\mathcal{A} \subset p\mathcal{M}p$ any von Neumann subalgebra. Assume that there exist $c > 0$ and $t \in (0, 1)$ such that $\mathrm{Tr}(\alpha_t(w)w^*) \geq c$ for all $w \in \mathcal{U}(\mathcal{A})$.*

Then there exists $q \in \mathrm{Proj_f}(\mathcal{B})$ such that $\mathcal{A} \preceq_{\mathcal{M}} q\mathcal{B}q$ or there exists $i \in \{1, 2\}$ and $q_i \in \mathrm{Proj_f}(\mathcal{M}_i)$ such that $\mathcal{N}_{p\mathcal{M}p}(\mathcal{A})'' \preceq_{\mathcal{M}} q_i \mathcal{M}_i q_i$.

Proof. By assumption, there exist $c > 0$ and $t \in (0, 1)$ such that $\mathrm{Tr}(\alpha_t(w)w^*) \geq c$ for all $w \in \mathcal{U}(\mathcal{A})$. Choose $r \in \mathbb{N}$ large enough such that $2^{-r} \leq t$. Then $\mathrm{Tr}(\alpha_{2^{-r}}(w)w^*) \geq c$ for all $w \in \mathcal{U}(\mathcal{A})$. So, we may assume that $t = 2^{-r}$. A standard functional analysis trick yields a nonzero partial isometry $v \in \alpha_t(p)\widetilde{\mathcal{M}}p$ such that $vx = \alpha_t(x)v$ for all $x \in \mathcal{A}$. Observe that $v^*v \in \mathcal{A}' \cap p\widetilde{\mathcal{M}}p$ and $vv^* \in \alpha_t(\mathcal{A}' \cap p\widetilde{\mathcal{M}}p)$.

We prove the result by contradiction. Using Proposition III.2.7 and as in the proof of Proposition III.2.4, we may choose a net of unitaries $w_k \in \mathcal{U}(\mathcal{A})$ such that $\lim_k \|E_{\mathcal{M}_i}(x^*w_ky)\|_{2,\mathrm{Tr}} = 0$ for all $i \in \{1,2\}$ and all $x, y \in p\mathcal{M}$. In particular, we get $\lim_k \|E_{\mathcal{B}}(x^*w_ky)\|_{2,\mathrm{Tr}} = 0$ for all $x, y \in p\mathcal{M}$. Regarding $\widetilde{\mathcal{M}} = \mathcal{M} *_{\mathcal{B}} (\mathcal{B}\overline{\otimes}\mathrm{L}(\mathbb{F}_2))$, we get $v^*v \in \mathcal{A}' \cap p\mathcal{M}p$ by Theorem III.2.5. We use now Popa's malleability trick [Po06a] and put $w = \alpha_t(v\beta(v^*)) \in \alpha_{2t}(p)\widetilde{\mathcal{M}}p$. Since $ww^* = \alpha_t(vv^*) \neq 0$, we get $w \neq 0$ and $wx = \alpha_{2t}(x)w$ for all $x \in \mathcal{A}$. Iterating this construction, we find a nonzero partial isometry $v \in \alpha_1(p)\widetilde{\mathcal{M}}p$ such that

$$vx = \alpha_1(x)v, \ \forall x \in \mathcal{A}. \tag{III.1}$$

Moreover, using again Proposition III.2.5, we get $v^*v \in \mathcal{A}' \cap p\mathcal{M}p$ and $vv^* \in \alpha_1(\mathcal{A}' \cap p\mathcal{M}p)$.

Next, exactly as in the proof of [CH10, Claim 4.3], we obtain the following.

Claim. We have $\lim_k \|E_{\alpha_1(\mathcal{M})}(x^*w_ky)\|_{2,\mathrm{Tr}} = 0$ for all $x, y \in p\widetilde{\mathcal{M}}$.

Proof of the Claim. Regard $\widetilde{\mathcal{M}} = \mathcal{M} *_{\mathcal{B}} (\mathcal{B} \overline{\otimes} \mathrm{L}(\mathbb{F}_2))$. By Kaplansky density theorem, it suffices to prove the Claim for $x = pa$ and $y = pb$ with a, b in $\mathcal{B}$ or reduced words in $\widetilde{\mathcal{M}}$ with letters alternating from $\mathcal{M} \ominus \mathcal{B}$ and $\mathcal{B} \overline{\otimes} \mathrm{L}(\mathbb{F}_2) \ominus \mathcal{B} \overline{\otimes} \mathbb{C}1$. Write $a = ca'$ with $c = a$ if $a \in \mathcal{B}$; $c = 1$ if a begins with a letter from $\mathcal{B} \overline{\otimes} \mathrm{L}(\mathbb{F}_2) \ominus \mathcal{B} \overline{\otimes} \mathbb{C}1$; c equals the first letter of a otherwise. Likewise, write $b = db'$. Then we have $x^*w_ky = a^*w_kb = a'^*\, c^*w_kd\, b'$ and note that $c^*w_kd \in \mathcal{M}$. Observe that a' (resp. b') equals 1 or is a reduced word beginning with a letter from $\mathcal{B} \overline{\otimes} \mathrm{L}(\mathbb{F}_2) \ominus \mathcal{B} \overline{\otimes} \mathbb{C}1$.

Denote by P the orthogonal projection from $\mathrm{L}^2(\mathcal{M}, \mathrm{Tr})$ onto $\mathcal{H}_0 \oplus \mathcal{H}_1$. Observe that since $c^*w_kd \in \mathcal{M} \cap \mathrm{L}^2(\mathcal{M}, \mathrm{Tr})$, we have

$$P(c^*w_kd) = E_{\mathcal{M}_1}(c^*w_kd) + E_{\mathcal{M}_2}(c^*w_kd) - E_{\mathcal{B}}(c^*w_kd).$$

Hence, $\lim_k \|P(c^*w_kd)\|_{2,\mathrm{Tr}} = 0$. Moreover, a simple calculation shows that

$$E_{\alpha_1(\mathcal{M})}(x^*w_ky) = E_{\alpha_1(\mathcal{M})}(a'^*P(c^*w_kd)b').$$

Therefore, $\lim_k \|E_{\alpha_1(\mathcal{M})}(x^*w_ky)\|_{2,\mathrm{Tr}} = 0$. This finishes the proof of the Claim. □

Finally, combining Equation $(III.1)$ together with the Claim, we get

$$\|vv^*\|_{2,\mathrm{Tr}} = \|\alpha_1(w_k)vv^*\|_{2,\mathrm{Tr}} = \|E_{\alpha_1(\mathcal{M})}(\alpha_1(w_k)vv^*)\|_{2,\mathrm{Tr}} = \|E_{\alpha_1(\mathcal{M})}(vw_kv^*)\|_{2,\mathrm{Tr}} \to 0.$$

This contradicts the fact that $v \neq 0$ and finishes the proof of Theorem III.3.3. □

III.3.3 A semifinite analogue of Ioana's intertwining theorem [Io(12)a]

Let $\mathcal{M} = \mathcal{M}_1 *_{\mathcal{B}} \mathcal{M}_2$ be a semifinite amalgamated free product von Neumann algebra with semifinite faithful normal trace Tr. Put $\widetilde{\mathcal{M}} = \mathcal{M} *_{\mathcal{B}} (\mathcal{B} \overline{\otimes} \mathrm{L}(\mathbb{F}_2))$ and observe that $\widetilde{\mathcal{M}}$ is still a semifinite amalgamated free product von Neumann algebra. We still denote by Tr the semifinite faithful normal trace on $\widetilde{\mathcal{M}}$. Let $\mathcal{N} = \bigvee\{u_g\mathcal{M}u_g^* : g \in \mathbb{F}_2\} \subset \widetilde{\mathcal{M}}$. Observe that $\mathcal{N}$ can be identified with an infinite amalgamated free product von Neumann algebra, that $\mathrm{Tr}\,|\mathcal{N}$ is semifinite and that, under this identification, the action $\mathbb{F}_2 \curvearrowright \mathcal{N}$ is given by the *free Bernoulli shift* which preserves the canonical trace Tr. We moreover have $\widetilde{\mathcal{M}} = \mathcal{N} \rtimes \mathbb{F}_2$.

We will denote by $E_{\mathcal{N}} : \widetilde{\mathcal{M}} \to \mathcal{N}$ the unique trace preserving faithful normal conditional expectation as well as the orthogonal projection $E_{\mathcal{N}} : \mathrm{L}^2(\widetilde{\mathcal{M}}, \mathrm{Tr}) \to \mathrm{L}^2(\mathcal{N}, \mathrm{Tr})$.

We prove next the analogue of [Io(12)a, Theorem 3.2] for semifinite amalgamated free product von Neumann algebras.

Theorem III.3.4. *Let $\mathcal{M} = \mathcal{M}_1 *_{\mathcal{B}} \mathcal{M}_2$ be a semifinite amalgamated free product von Neumann algebra with semifinite faithful normal trace* Tr. *Let $p \in \mathrm{Proj_f}(\mathcal{B})$, $\mathcal{A} \subset p\mathcal{M}p$ any von Neumann subalgebra and $t \in (0,1)$. Assume that there is no net of unitaries $w_k \in \mathcal{U}(\mathcal{A})$ such that*

$$\lim_k \|E_{\mathcal{N}}(x^* \alpha_t(w_k) y)\|_{2,\mathrm{Tr}} = 0, \, \forall x, y \in p\widetilde{\mathcal{M}}.$$

Then there exists $q \in \mathrm{Proj_f}(\mathcal{B})$ such that $\mathcal{A} \preceq_{\mathcal{M}} q\mathcal{B}q$ or there exists $i \in \{1,2\}$ and $q_i \in \mathrm{Proj_f}(\mathcal{M}_i)$ such that $\mathcal{N}_{p\mathcal{M}p}(\mathcal{A})'' \preceq_{\mathcal{M}} q_i\mathcal{M}_i q_i$.

The main technical lemma that will be used to prove Theorem III.3.4 is a straightforward generalization of [Io(12)a, Lemma 3.4]. We include a proof for the sake of completeness.

Lemma III.3.5. *Let $t \in (0,1)$ and $g, h \in \mathbb{F}_2$. For all $n \geq 0$, define*

$$c_n = \sup_{x \in \mathcal{H}_n, \, \|x\|_{2,\mathrm{Tr}} \leq 1} \|E_{\mathcal{N}}(u_g \alpha_t(x) u_h)\|_{2,\mathrm{Tr}}.$$

Then $\lim_n c_n = 0$.

Proof. First, observe that for all $g_1 \dots, g_{n+1} \in \mathbb{F}_2$ and all $x_1, \dots, x_n \in \mathcal{M}$, we have

$$E_{\mathcal{N}}(u_{g_1} x_1 \cdots u_{g_n} x_n u_{g_{n+1}}) = \begin{cases} u_{g_1} x_1 \cdots u_{g_n} x_n u_{g_{n+1}} \text{ if } g_1 \cdots g_{n+1} = 1; \\ 0 \text{ otherwise.} \end{cases} \tag{III.2}$$

Thus for all $\mathcal{I} \in S_n$, we have $E_{\mathcal{N}}(\mathcal{K}_{\mathcal{I}}) \subset \mathcal{K}_{\mathcal{I}}$ and since $\alpha_t(\mathcal{H}_{\mathcal{I}}) \subset \mathcal{K}_{\mathcal{I}}$, we get that $E_{\mathcal{N}}(u_g \alpha_t(x) u_h) \in \mathcal{K}_{\mathcal{I}}$ for all $x \in \mathcal{H}_{\mathcal{I}}$. So defining

$$c_{\mathcal{I}} = \sup_{x \in \mathcal{H}_{\mathcal{I}}, \, \|x\|_{2,\mathrm{Tr}} \leq 1} \|E_{\mathcal{N}}(u_g \alpha_t(x) u_h)\|_{2,\mathrm{Tr}},$$

we see that $c_n = \max_{\mathcal{I} \in S_n} c_{\mathcal{I}}$ since the subspaces $\mathcal{K}_{\mathcal{I}}$'s are pairwise orthogonal.

Let us fix $\mathcal{I} = (i_1, \dots, i_n) \in S_n$ and calculate $c_{\mathcal{I}}$. Denote by a and b the canonical generators of $\mathbb{F}_2$ so that $u_1 = u_a$, $u_2 = u_b$ and put $G_1 = \langle a \rangle$ and $G_2 = \langle b \rangle$. For $g_1, h_1 \in G_{i_1}, \dots, g_n, h_n \in G_{i_n}$, define a map

$$V_{g_1,h_1,\dots,g_n,h_n}(x_1 \cdots x_n) = u_{g_1} x_1 u_{h_1}^* \cdots u_{g_n} x_n u_{h_n}^*$$

for all $x_j \in \mathcal{M}_{i_j} \ominus \mathcal{B}$ such that $\mathrm{Tr}(x_j^* x_j) < \infty$ for all $j \in \{1, \dots, n\}$. By Lemma III.3.2, these maps $V_{g_1,h_1,\dots,g_n,h_n}$ extend to isometries $V_{g_1,h_1,\dots,g_n,h_n} : \mathcal{H}_{\mathcal{I}} \to \mathcal{K}_{\mathcal{I}}$ with pairwise orthogonal ranges when $(g_1, h_1, \dots, g_n, h_n)$ are pairwise distinct. Indeed, we have $V_{g_1,h_1,\dots,g_n,h_n}(\mathcal{H}_{\mathcal{I}}) \perp V_{g_1',h_1',\dots,g_n',h_n'}(\mathcal{H}_{\mathcal{I}})$ unless $g_1 = g_1', h_1^{-1} g_2 = h_1'^{-1} g_2', \dots, h_{n-1}^{-1} g_n = h_{n-1}'^{-1} g_n', h_n^{-1} = h_n'^{-1}$. Since moreover $G_1 \cap G_2 = \{e\}$, this further implies that $g_j = g_j'$ and $h_j = h_j'$ for all $j \in \{1, \dots, n\}$.

Denote the Fourier coefficients of u_1^t and u_2^t respectively by $\beta_1(g_1) = \tau(u_1^t u_{g_1}^*)$ for $g_1 \in G_1$ and $\beta_2(g_2) = \tau(u_2^t u_{g_2}^*)$ for $g_2 \in G_2$. We have an explicit formula for these coefficients given by

$$\beta_i(u_i^n) = \tau(u_i^t u_i^{-n}) = \tau(u_i^{t-n}) = \frac{\sin(\pi(t-n))}{\pi(t-n)}.$$

It follows in particular that $\beta_i(g_i) \in \mathbb{R}$ for all $i \in \{1,2\}$ and all $g_i \in G_i$. Since u_1^t and u_2^t are unitaries, we moreover have

$$\sum_{g_1 \in G_1} \beta_1(g_1)^2 = \sum_{g_2 \in G_2} \beta_2(g_2)^2 = 1.$$

If $x = x_1 \cdots x_n$ with $x_j \in \mathcal{M}_{i_j} \ominus \mathcal{B}$ satisfying $\mathrm{Tr}(x_j^* x_j) < \infty$, we have

$$\begin{aligned} u_g \alpha_t(x) u_h &= u_g\, u_{i_1}^t x_1 u_{i_1}^{t*} \cdots u_{i_n}^t x_n u_{i_n}^{t*}\, u_h \\ &= \sum_{g_1,h_1 \in G_{i_1},\ldots,g_n,h_n \in G_{i_n}} \beta_{i_1}(g_1)\beta_{i_1}(h_1)\cdots\beta_{i_n}(g_n)\beta_{i_n}(h_n)\; u_g\, u_{g_1} x_1 u_{h_1}^* \cdots u_{g_n} x_n u_{h_n}^*\, u_h \\ &= \sum_{g_1,h_1 \in G_{i_1},\ldots,g_n,h_n \in G_{i_n}} \beta_{i_1}(g_1)\beta_{i_1}(h_1)\cdots\beta_{i_n}(g_n)\beta_{i_n}(h_n)\; u_g V_{g_1,h_1,\ldots,g_n,h_n}(x) u_h, \end{aligned}$$

where the sum converges in $\|\cdot\|_{2,\mathrm{Tr}}$. Thus, for all $x \in \mathcal{H}_{\mathcal{I}}$, we get

$$u_g \alpha_t(x) u_h = \sum_{g_1,h_1 \in G_{i_1},\ldots,g_n,h_n \in G_{i_n}} \beta_{i_1}(g_1)\beta_{i_1}(h_1)\cdots\beta_{i_n}(g_n)\beta_{i_n}(h_n)\; u_g V_{g_1,h_1,\ldots,g_n,h_n}(x) u_h.$$

Now, using the calculation $(III.2)$, and the fact that the isometries $V_{g_1,h_1,\ldots,g_n,h_n}$ have mutually orthogonal ranges, we get that for all $x \in \mathcal{H}_{\mathcal{I}}$,

$$\|E_{\mathcal{N}}(u_g \alpha_t(x) u_h)\|_{2,\mathrm{Tr}}^2 = \|x\|_{2,\mathrm{Tr}}^2 \sum_{\substack{g_1,h_1 \in G_{i_1},\ldots,g_n,h_n \in G_{i_n} \\ g g_1 h_1^{-1} \cdots g_n h_n^{-1} h = 1}} \beta_{i_1}(g_1)^2 \beta_{i_1}(h_1)^2 \cdots \beta_{i_n}(g_n)^2 \beta_{i_n}(h_n)^2.$$

Thus we get an explicit formula for $c_{\mathcal{I}}$ given by

$$c_{\mathcal{I}} = \sum_{\substack{g_1,h_1 \in G_{i_1},\ldots,g_n,h_n \in G_{i_n} \\ g g_1 h_1 \cdots g_n h_n h = 1}} \beta_{i_1}(g_1)^2 \beta_{i_1}(h_1^{-1})^2 \ldots \beta_{i_n}(g_n)^2 \beta_{i_n}(h_n^{-1})^2. \tag{III.3}$$

For $i \in \{1,2\}$, define $\mu_i \in \mathrm{Prob}(\mathbb{F}_2)$ by $\mu_i(g) = \beta_i(g)^2$ if $g \in G_i$ and $\mu_i(g) = 0$ otherwise. Likewise, define $\check{\mu}_i \in \mathrm{Prob}(\mathbb{F}_2)$ by $\check{\mu}_i(g) = \mu_i(g^{-1})$ for all $g \in \mathbb{F}_2$. Put $\nu_i = \mu_i * \check{\mu}_i$. Then we have

$$c_{\mathcal{I}} = (\nu_{i_1} * \cdots * \nu_{i_n})(g^{-1}h^{-1}).$$

So if we put $\mu = \nu_1 * \nu_2$, we have that

$$c_{\mathcal{I}} \in \left\{ \mu^{*[\frac{n}{2}]}(g^{-1}h^{-1}), \mu^{*[\frac{n}{2}]} * \nu_1(g^{-1}h^{-1}), \nu_2 * \mu^{*[\frac{n}{2}]}(g^{-1}h^{-1}), \nu_2 * \mu^{*[\frac{n-1}{2}]} * \nu_1(g^{-1}h^{-1}) \right\}.$$

Then [Io(12)a, Lemma 2.13] implies that $\lim_k \mu^{*k}(s) = 0$ for all $s \in \mathbb{F}_2$ and so $\lim_n c_n = 0$. $\square$

Proof of Theorem III.3.4. Assume by contradiction that the conclusion of the theorem does not hold. Then Theorem III.3.3 implies that for $t \in (0,1)$ there exists a net $w_k \in \mathcal{U}(\mathcal{A})$ such that

$$\lim_k \mathrm{Tr}(\alpha_t(w_k) w_k^*) = 0.$$

We will show that for all $x, y \in p\widetilde{\mathcal{M}}$, we have $\lim_k \|E_{\mathcal{N}}(x^*\alpha_t(w_k)y)\|_{2,\mathrm{Tr}} = 0$, which will contradict the assumption of Theorem III.3.4. By a linearity/density argument, it is sufficient to show that for all $g, h \in \mathbb{F}_2$,

$$\lim_k \|E_{\mathcal{N}}(u_g\alpha_t(w_k)u_h)\|_{2,\mathrm{Tr}} = 0. \tag{III.4}$$

For all k, we have $w_k \in \mathcal{A} \subset \mathrm{L}^2(\mathcal{M}, \mathrm{Tr}) = \bigoplus_{n\in\mathbb{N}} \mathcal{H}_n$ so that we can write $w_k = \sum_{n\in\mathbb{N}} w_{k,n}$, with $w_{k,n} \in \mathcal{H}_n$. Recall that $\mathrm{Tr}(\alpha_t(w_k)w_k^*) = \sum_{n\in\mathbb{N}} \left(\frac{\sin(\pi t)}{\pi t}\right)^{2n} \|w_{k,n}\|^2_{2,\mathrm{Tr}}$. Thus the fact that $\lim_k \mathrm{Tr}(\alpha_t(w_k)w_k^*) = 0$ implies that for all $n \geq 0$, $\lim_k \|w_{k,n}\|_{2,\mathrm{Tr}} = 0$.

Fix $g, h \in \mathbb{F}_2$ and $\varepsilon > 0$. Note that for $n \geq 1$, $E_{\mathcal{N}}(u_g\alpha_t(w_{k,n})u_h) \in \mathcal{K}_n$, so that all these terms are pairwise orthogonal. They are also all orthogonal to $E_{\mathcal{N}}(u_g\alpha_t(w_{k,0})u_h)$, which belongs to $\mathcal{K}_0$. Thus

$$\begin{aligned}\|E_{\mathcal{N}}(u_g\alpha_t(w_k)u_h)\|^2_{2,\mathrm{Tr}} &= \sum_{n\geq 0} \|E_{\mathcal{N}}(u_g\alpha_t(w_{k,n})u_h)\|^2_{2,\mathrm{Tr}} \\ &\leq \sum_{n\geq 0} c_n^2 \|w_{k,n}\|^2_{2,\mathrm{Tr}}\end{aligned}$$

where c_n is defined in Lemma III.3.5. Observe that $c_n \leq 1$ for all $n \in \mathbb{N}$.

Lemma III.3.5 implies that there exists $n_0 \geq 0$ such that for all $n > n_0$, $c_n^2 < \varepsilon/2$. Then we can find k_0 such that for all $k \geq k_0$, and all $n \leq n_0$, $\|w_{k,n}\|^2_{2,\mathrm{Tr}} < \varepsilon/2(n_0+1)$. So we get that for all $k \geq k_0$,

$$\|E_{\mathcal{N}}(u_g\alpha_t(w_k)u_h)\|^2_{2,\mathrm{Tr}} \leq \sum_{n=0}^{n_0} \|w_{k,n}\|^2_{2,\mathrm{Tr}} + \frac{\varepsilon}{2} \sum_{n\geq n_0} \|w_{k,n}\|^2_{2,\mathrm{Tr}} \leq \sum_{n=0}^{n_0} \|w_{k,n}\|^2_{2,\mathrm{Tr}} + \frac{\varepsilon}{2}\|w_k\|^2_{2,\mathrm{Tr}} \leq \varepsilon.$$

This shows $(III.4)$ and finishes the proof of Theorem III.3.4. □

III.4 Relative amenability inside semifinite AFP von Neumann algebras

Let $\mathcal{M} = \mathcal{M}_1 *_{\mathcal{B}} \mathcal{M}_2$ be a semifinite amalgamated free product von Neumann algebra with semifinite faithful normal trace Tr. Recall that $\widetilde{\mathcal{M}} = \mathcal{M} *_{\mathcal{B}} (\mathcal{B} \mathbin{\overline{\otimes}} \mathrm{L}(\mathbb{F}_2))$, $\mathcal{N} = \bigvee\{u_g\mathcal{M}u_g^* : g \in \mathbb{F}_2\} \subset \widetilde{\mathcal{M}}$ and observe that $\widetilde{\mathcal{M}} = \mathcal{N} \rtimes \mathbb{F}_2$. We denote by $\alpha : \mathbb{R} \to \mathrm{Aut}(\widetilde{\mathcal{M}})$ the malleable deformation from Section III.3.1.

The main result of this section is the following strengthening of Ioana's result [Io(12)a, Theorem 4.1] in the framework of semifinite amalgamated free product von Neumann algebras over an *amenable* subalgebra.

Theorem III.4.1. *Let $\mathcal{M} = \mathcal{M}_1 *_{\mathcal{B}} \mathcal{M}_2$ be a semifinite amalgamated free product von Neumann algebra with semifinite faithful normal trace $\mathrm{Tr}_{\mathcal{M}}$. Assume that $\mathcal{B}$ is amenable. Let $q \in \mathrm{Proj_f}(\mathcal{B})$ such that $q\mathcal{M}_1q \neq q\mathcal{B}q \neq q\mathcal{M}_2q$ and $t \in (0,1)$ such that $\alpha_t(q\mathcal{M}q)$ is amenable relative to $q\mathcal{N}q$ inside $q\widetilde{\mathcal{M}}q$.*

Then for all $i \in \{1, 2\}$, there exists a nonzero projection $z_i \in \mathcal{Z}(\mathcal{M}_i)$ such that $\mathcal{M}_iz_i$ is amenable.

Let $\mathrm{Tr}_{\widetilde{\mathcal{M}}}$ be the semifinite faithful normal trace on $\widetilde{\mathcal{M}} = \mathcal{M} *_{\mathcal{B}} (\mathcal{B} \mathbin{\overline{\otimes}} \mathrm{L}(\mathbb{F}_2))$. Consider the basic construction $\langle q\widetilde{\mathcal{M}}q, e_{q\mathcal{N}q}\rangle$ associated with the inclusion of tracial von Neumann algebras $q\mathcal{N}q \subset q\widetilde{\mathcal{M}}q$.

We denote by $\tau = \frac{1}{\mathrm{Tr}_{\widetilde{\mathcal{M}}}(q)} \mathrm{Tr}_{\widetilde{\mathcal{M}}}(q \cdot q)$ the faithful normal tracial state on $q\widetilde{\mathcal{M}}q$ and by $\| \cdot \|_2$ the L^2-norm on $q\widetilde{\mathcal{M}}q$ associated with τ. We then simply denote by Tr the canonical semifinite faithful normal trace on $\langle q\widetilde{\mathcal{M}}q, e_{q\mathcal{N}q}\rangle$ given by $\mathrm{Tr}(ae_{q\mathcal{N}q}b) = \tau(ab)$ for all $a, b \in q\widetilde{\mathcal{M}}q$. Observe that $q\widetilde{\mathcal{M}}q = q\mathcal{N}q \rtimes \mathbb{F}_2$. Following [Io(12)a, Section 4], we define the $q\mathcal{M}_1q$-$q\mathcal{M}_1q$-bimodule

$$\mathcal{H}_1 = \bigoplus_{g\in\mathbb{F}_2} \mathrm{L}^2(q\mathcal{M}_1q)\, u_g e_{q\mathcal{N}q} u_g^* \subset \mathrm{L}^2(\langle q\widetilde{\mathcal{M}}q, e_{q\mathcal{N}q}\rangle).$$

Denote by $\mathcal{H} = \mathrm{L}^2(\langle q\widetilde{\mathcal{M}}q, e_{q\mathcal{N}q}\rangle, \mathrm{Tr}) \ominus \mathcal{H}_1$.

Lemma III.4.2. *As $q\mathcal{M}_1q$-$q\mathcal{M}_1q$-bimodules, we have that $\mathcal{H} \subset_{\mathrm{weak}} \mathrm{L}^2(q\mathcal{M}_1q) \otimes \mathrm{L}^2(q\mathcal{M}_1q)$.*

Proof. The proof goes along the same lines as [Io(12)a, Lemma 4.2]. First observe that since $q\widetilde{\mathcal{M}}q = q\mathcal{N}q \rtimes \mathbb{F}_2$, we have

$$\mathrm{L}^2(\langle q\mathcal{M}q, e_{q\mathcal{N}q}\rangle) \cong \bigoplus_{g,h\in\mathbb{F}_2} \mathrm{L}^2(q\mathcal{N}q)\, u_g e_{q\mathcal{N}q} u_h.$$

So it suffices to prove that for all $g, h \in \mathbb{F}_2$ such that $h \neq g^{-1}$, as $q\mathcal{M}_1q$-$q\mathcal{M}_1q$-bimodules, we have

$$\begin{aligned}\big(\mathrm{L}^2(q\mathcal{N}q) \ominus \mathrm{L}^2(q\mathcal{M}_1q)\big)\, u_g e_{q\mathcal{N}q} u_g^* &\subset_{\mathrm{weak}} \mathrm{L}^2(q\mathcal{M}_1q) \otimes \mathrm{L}^2(q\mathcal{M}_1q)\\ \mathrm{L}^2(q\mathcal{N}q)\, u_g e_{q\mathcal{N}q} u_h &\subset_{\mathrm{weak}} \mathrm{L}^2(q\mathcal{M}_1q) \otimes \mathrm{L}^2(q\mathcal{M}_1q).\end{aligned}$$

Denote by $\mathrm{L}^2(q\mathcal{N}q)^g$ the $q\mathcal{M}_1q$-$q\mathcal{M}_1q$-bimodule $\mathrm{L}^2(q\mathcal{N}q)$ with left and right action given by $x \cdot \xi \cdot y = x\xi u_g y u_g^*$ for all $x, y \in q\mathcal{M}_1q$ and all $\xi \in \mathrm{L}^2(q\mathcal{N}q)$. Likewise, define the $\mathcal{M}_1$-$\mathcal{M}_1$-bimodule $\mathrm{L}^2(\mathcal{N})^g$. As $q\mathcal{M}_1q$-$q\mathcal{M}_1q$-bimodules, we have

$$\begin{aligned}\bigoplus_{g\in\mathbb{F}_2} \big(\mathrm{L}^2(q\mathcal{N}q) \ominus \mathrm{L}^2(q\mathcal{M}_1q)\big)\, u_g e_{q\mathcal{N}q} u_g^* &\cong \bigoplus_{i=1}^{\infty} \big(\mathrm{L}^2(q\mathcal{N}q) \ominus \mathrm{L}^2(q\mathcal{M}_1q)\big)\\ \bigoplus_{g,h\in\mathbb{F}_2, h\neq g^{-1}} \mathrm{L}^2(q\mathcal{N}q)\, u_g e_{q\mathcal{N}q} u_h &\cong \bigoplus_{i=1}^{\infty} \bigoplus_{g\in\mathbb{F}_2\setminus\{e\}} \mathrm{L}^2(q\mathcal{N}q)^g.\end{aligned}$$

Put $\mathcal{P} = \Big(\bigcup_{h\in\mathbb{F}_2\setminus\{e\}} u_h\mathcal{M}u_h^* \cup \mathcal{M}_2\Big)''$ and $\mathcal{P}_g = \Big(\bigcup_{h\in\mathbb{F}_2\setminus\{e,g\}} u_h\mathcal{M}u_h^* \cup \mathcal{M}_2 \cup u_g\mathcal{M}_2u_g^*\Big)''$ for all $g \in \mathbb{F}_2$. Then we have

$$\mathcal{N} \cong \mathcal{M}_1 *_{\mathcal{B}} \mathcal{P} \cong \mathcal{M}_1 *_{\mathcal{B}} u_g\mathcal{M}_1u_g^* *_{\mathcal{B}} \mathcal{P}_g, \forall g \in \mathbb{F}_2 \setminus \{e\}.$$

Using [Ue99, Section 2], there are $\mathcal{B}$-$\mathcal{B}$-bimodules $\mathcal{L}$ and $\mathcal{L}_g$ for $g \in \mathbb{F}_2 \setminus \{e\}$, such that as $\mathcal{M}_1$-$\mathcal{M}_1$-bimodules, we have

$$\begin{aligned}\mathrm{L}^2(\mathcal{N}) \ominus \mathrm{L}(\mathcal{M}_1) &\cong \mathrm{L}^2(\mathcal{M}_1) \otimes_{\mathcal{B}} \mathcal{L} \otimes_{\mathcal{B}} \mathrm{L}^2(\mathcal{M}_1)\\ \mathrm{L}^2(\mathcal{N})^g &\cong \mathrm{L}^2(\mathcal{M}_1) \otimes_{\mathcal{B}} \mathcal{L}_g \otimes_{\mathcal{B}} \mathrm{L}^2(\mathcal{M}_1).\end{aligned}$$

Since $\mathcal{B}$ is amenable, we have that $\mathrm{L}^2(\mathcal{B}) \subset_{\text{weak}} \mathrm{L}^2(\mathcal{B}) \otimes \mathrm{L}^2(\mathcal{B})$ as $\mathcal{B}$-$\mathcal{B}$-bimodules. Using [AD95, Lemma 1.7], we obtain that, as $q\mathcal{M}_1q$-$q\mathcal{M}_1q$-bimodules,

$$\begin{aligned}\mathrm{L}^2(q\mathcal{N}q) \ominus \mathrm{L}^2(q\mathcal{M}_1q) &\cong q\left(\mathrm{L}^2(\mathcal{M}_1) \otimes_{\mathcal{B}} \mathcal{L} \otimes_{\mathcal{B}} \mathrm{L}^2(\mathcal{M}_1)\right) q \\ &\subset_{\text{weak}} q\left(\mathrm{L}^2(\mathcal{M}_1) \otimes \mathcal{L} \otimes \mathrm{L}^2(\mathcal{M}_1)\right) q \\ &\subset_{\text{weak}} q\left(\mathrm{L}^2(\mathcal{M}_1) \otimes \mathrm{L}^2(\mathcal{M}_1)\right) q.\end{aligned}$$

Since $q\left(\mathrm{L}^2(\mathcal{M}_1) \otimes \mathrm{L}^2(\mathcal{M}_1)\right) q$ is isomorphic to a $q\mathcal{M}_1q$-$q\mathcal{M}_1q$-subbimodule of $\bigoplus_{i=1}^{\infty} \mathrm{L}^2(q\mathcal{M}_1q) \otimes \mathrm{L}^2(q\mathcal{M}_1q)$, we infer that, as $q\mathcal{M}_1q$-$q\mathcal{M}_1q$-bimodules,

$$\mathrm{L}^2(q\mathcal{N}q) \ominus \mathrm{L}^2(q\mathcal{M}_1q) \subset_{\text{weak}} \mathrm{L}^2(q\mathcal{M}_1q) \otimes \mathrm{L}^2(q\mathcal{M}_1q).$$

Similarly, for all $g \in \mathbb{F}_2 \setminus \{e\}$ we get that, as $q\mathcal{M}_1q$-$q\mathcal{M}_1q$-bimodules,

$$\mathrm{L}^2(q\mathcal{N}q)^g \subset_{\text{weak}} \mathrm{L}^2(q\mathcal{M}_1q) \otimes \mathrm{L}^2(q\mathcal{M}_1q). \qquad \square$$

Proof of Theorem III.4.1. Since $\alpha_t(q\mathcal{M}q)$ is amenable relative to $q\mathcal{N}q$ inside $q\widetilde{\mathcal{M}}q$, we find a net of vectors $\xi_n \in \mathrm{L}^2(\langle q\widetilde{\mathcal{M}}q, e_{q\mathcal{N}q}\rangle, \mathrm{Tr})$ for $n \in I$, such that

- $\langle x\xi_n \,|\, \xi_n\rangle_{\mathrm{Tr}} \to \tau(x)$ for all $x \in q\widetilde{\mathcal{M}}q$, and
- $\|x\xi_n - \xi_n x\|_{2,\mathrm{Tr}} \to 0$ for all $x \in \alpha_t(q\mathcal{M}q)$.

Observe that using the proof of [OP10a, Theorem 2.1] we may assume that $\xi_n \geq 0$ so that $\langle x\xi_n \,|\, \xi_n\rangle_{\mathrm{Tr}} = \mathrm{Tr}(x\xi_n^2) = \langle \xi_n x \,|\, \xi_n\rangle_{\mathrm{Tr}}$ for all $x \in q\widetilde{\mathcal{M}}q$ and all $n \in I$. Since $\|\xi_n\|_{2,\mathrm{Tr}} \to 1$, we may further assume that $\|\xi_n\|_{2,\mathrm{Tr}} = 1$ for all $n \in I$.

By contradiction, assume that for some $i \in \{1, 2\}$, $q\mathcal{M}_iq$ has no amenable direct summand. Without loss of generality, we may assume that $i = 1$. Denote by $P_{\mathcal{H}_1} : \mathrm{L}^2(\langle q\widetilde{\mathcal{M}}q, e_{q\mathcal{N}q}\rangle) \to \mathcal{H}_1$ the orthogonal projection. Observe that $P_{\mathcal{H}_1}$ is the orthogonal projection corresponding to the unique trace preserving faithful normal conditional expectation $E_{\mathcal{Q}} : q\widetilde{\mathcal{M}}q \to \mathcal{Q}$ onto the von Neumann subalgebra $\mathcal{Q} = \bigvee\{q\mathcal{M}_1q, u_g e_{q\mathcal{N}q} u_g^* : g \in \mathbf{F}_2\} \subset q\widetilde{\mathcal{M}}q$. We claim that $\lim_n \|u_1^{t*}\xi_n u_1^t - P_{\mathcal{H}_1}(u_1^{t*}\xi_n u_1^t)\|_{2,\mathrm{Tr}} = 0$. If this is not the case, let $\zeta_n = (1 - P_{\mathcal{H}_1})(u_1^{t*}\xi_n u_1^t) \in \mathcal{H}$ and observe that $\limsup_n \|\zeta_n\|_{2,\mathrm{Tr}} > 0$. Arguing as in the proof of [Io(12)a, Lemma 2.3], we may further assume that $\liminf_n \|\zeta_n\|_{2,\mathrm{Tr}} > 0$.

Then $\zeta_n \in \mathcal{H}$ is a net of vectors which satisfies the following conditions:

- $\liminf_n \|\zeta_n\|_{2,\mathrm{Tr}} > 0$;
- $\limsup_n \|x\zeta_n\|_{2,\mathrm{Tr}} \leq \|x\|_2$ for all $x \in q\mathcal{M}_1q$;
- $\lim_n \|y\zeta_n - \zeta_n y\|_{2,\mathrm{Tr}} = 0$ for all $y \in q\mathcal{M}_1q$.

Since as $q\mathcal{M}_1q$-$q\mathcal{M}_1q$-bimodules, we have that $\mathcal{H} \subset_{\text{weak}} \mathrm{L}^2(q\mathcal{M}_1q) \otimes \mathrm{L}^2(q\mathcal{M}_1q)$ by Lemma III.4.2, it follows that $q\mathcal{M}_1q$ has an amenable direct summand by Connes's result [Co76]. This contradicts our assumption and we have shown that $\lim_n \|\xi_n - u_1^t P_{\mathcal{H}_1}(u_1^{t*}\xi_n u_1^t)u_1^{t*}\|_{2,\mathrm{Tr}} = \lim_n \|u_1^{t*}\xi_n u_1^t - P_{\mathcal{H}_1}(u_1^{t*}\xi_n u_1^t)\|_{2,\mathrm{Tr}} = 0$.

Put $\mathcal{L}_1 = u_1^t \mathcal{H}_1 u_1^{t*}$ and denote by $P_{\mathcal{L}_1} : \mathrm{L}^2(\langle q\widetilde{\mathcal{M}}q, e_{q\mathcal{N}q}\rangle) \to \mathcal{L}_1$ the orthogonal projection. Put $\eta_n = P_{\mathcal{L}_1}(\xi_n)$ and observe that $\eta_n \in \mathcal{L}_1$ and $\eta_n \geq 0$. We moreover have $\lim_n \|\xi_n - \eta_n\|_{2,\mathrm{Tr}} = 0$. So $\eta_n \in \mathcal{L}_1$ is a net of vectors which satisfy

$(*)$ $\langle x\eta_n \,|\, \eta_n\rangle_{\mathrm{Tr}} = \langle \eta_n x \,|\, \eta_n\rangle_{\mathrm{Tr}} \to \tau(x)$ for all $x \in q\widetilde{\mathcal{M}}q$, and

$(**)$ $\|x\eta_n - \eta_n x\|_{2,\mathrm{Tr}} \to 0$ for all $x \in \alpha_t(q\mathcal{M}q)$.

We have $\eta_n = \sum_{g\in\mathbb{F}_2} u_1^t x_{n,g} u_g e_{q\mathcal{N}q} u_g^* u_1^{t*}$ with $x_{n,g} \in \mathrm{L}^2(q\mathcal{M}_1 q)$. Since $\eta_n = \eta_n^*$ for all $n \in I$, we may assume that $x_{n,g} = x_{n,g}^*$ for all $n \in I$ and all $g \in \mathbb{F}_2$. Next, we claim that we may further assume that $x_{n,g} \in q\mathcal{M}_1 q$ with $x_{n,g} = x_{n,g}^*$ for all $n \in I$ and all $g \in \mathbb{F}_2$.

To do so, define the set J of triples $j = (X, Y, \varepsilon)$ where $X \subset \mathrm{Ball}(q\widetilde{\mathcal{M}}q)$, $Y \subset \mathrm{Ball}(\alpha_t(q\mathcal{M}q))$ are finite subsets and $\varepsilon > 0$. We make J a directed set by putting $(X, Y, \varepsilon) \leq (X', Y', \varepsilon')$ if and only if $X \subset X'$, $Y \subset Y'$ and $\varepsilon' \leq \varepsilon$. Let $j = (X, Y, \varepsilon) \in J$. There exists $n \in I$ such that $|\langle x\eta_n \,|\, \eta_n\rangle_{\mathrm{Tr}} - \tau(x)| \leq \varepsilon/2$ and $\|y\eta_n - \eta_n y\|_{2,\mathrm{Tr}} \leq \varepsilon/2$ for all $x \in X$ and all $y \in Y$. Let $v \in \ell^2(\mathbb{F}_2)_+$ such that $\|v\|_{\ell^2(\mathbb{F}_2)} = 1$. For each $g \in \mathbb{F}_2$, choose $y_{j,g} \in q\mathcal{M}_1 q$ such that $y_{j,g} = y_{j,g}^*$ and $\|x_{n,g} - y_{j,g}\|_2 \leq v(g)\,\varepsilon/4$. Put $\eta_j' = \sum_{g\in\mathbb{F}_2} u_1^t y_{j,g} u_g e_{q\mathcal{N}q} u_g^* u_1^{t*} \in \mathcal{L}_1$ and observe that $\eta_j' = \eta_j'^*$ and $\|\eta_n - \eta_j'\|_{2,\mathrm{Tr}} \leq \varepsilon/4$. We get $|\langle x\eta_j' \,|\, \eta_j'\rangle_{\mathrm{Tr}} - \tau(x)| \leq \varepsilon + \varepsilon^2/16$ and $\|y\eta_j' - \eta_j' y\|_{2,\mathrm{Tr}} \leq \varepsilon$ for all $x \in X$ and all $y \in Y$. Then the net $(\eta_j')_{j\in J}$ clearly satisfies Conditions $(*)$ and $(**)$ above. This finishes the proof of the claim.

Fix any $y \in q\mathcal{M}_2 q \ominus q\mathcal{B}q$ satisfying $\|y\|_2 = 1$. Then we have

$$\langle \alpha_t(y)\eta_n \,|\, \eta_n\alpha_t(y)\rangle_{\mathrm{Tr}} \to 1.$$

Expanding $\alpha_t(y)$ and η_n, we obtain

$$\begin{aligned}
\langle \alpha_t(y)\eta_n \,|\, \eta_n\alpha_t(y)\rangle_{\mathrm{Tr}} &= \sum_{g,h\in\mathbb{F}_2} \left\langle u_2^t y u_2^{t*}\, u_1^t x_{n,g} u_g e_{q\mathcal{N}q} u_g^* u_1^{t*} \,|\, u_1^t x_{n,h} u_h e_{q\mathcal{N}q} u_h^* u_1^{t*}\, u_2^t y u_2^{t*} \right\rangle_{\mathrm{Tr}} \\
&= \sum_{g,h\in\mathbb{F}_2} \left\langle u_h^* x_{n,h}^* u_1^{t*} u_2^t y u_2^{t*} u_1^t x_{n,g} u_g\, e_{q\mathcal{N}q} \,|\, e_{q\mathcal{N}q}\, u_h^* u_1^{t*} u_2^t y u_2^{t*} u_1^t u_g \right\rangle_{\mathrm{Tr}} \\
&= \sum_{g,h\in\mathbb{F}_2} \tau\left(E_{q\mathcal{N}q}(u_g^* u_1^{t*} u_2^t y^* u_2^{t*} u_1^t u_h)\, E_{q\mathcal{N}q}(u_h^* x_{n,h}^* u_1^{t*} u_2^t y u_2^{t*} u_1^t x_{n,g} u_g) \right).
\end{aligned}$$

Recall from Section III.3.1 the definition of the Hilbert spaces $\mathcal{K}_k$ for $k \in \mathbb{N}$ and denote by $b_{n,g} = E_{q\mathcal{B}q}(x_{n,g})$. Since we have

$$E_{q\mathcal{N}q}\left(u_g^* u_1^{t*} u_2^t y^* u_2^{t*} u_1^t u_h\right) \in \mathcal{K}_1,$$

$$E_{q\mathcal{N}q}\left(u_h^* (x_{n,h} - b_{n,g})^* u_1^{t*} u_2^t y u_2^{t*} u_1^t b_{n,g} u_g\right) \text{ and } E_{q\mathcal{N}q}\left(u_h^* b_{n,g}^* u_1^{t*} u_2^t y u_2^{t*} u_1^t (x_{n,g} - b_{n,g}) u_g\right) \in \mathcal{K}_2,$$

$$E_{q\mathcal{N}q}\left(u_h^* (x_{n,h} - b_{n,g})^* u_1^{t*} u_2^t y u_2^{t*} u_1^t (x_{n,g} - b_{n,g}) u_g\right) \in \mathcal{K}_3,$$

we get

$$\begin{aligned}
\langle \alpha_t(y)\eta_n \,|\, \eta_n\alpha_t(y)\rangle_{\mathrm{Tr}} &= \sum_{g,h\in\mathbb{F}_2} \tau\left(E_{q\mathcal{N}q}(u_g^* u_1^{t*} u_2^t y^* u_2^{t*} u_1^t u_h)\, E_{q\mathcal{N}q}(u_h^* b_{n,h}^* u_1^{t*} u_2^t y u_2^{t*} u_1^t b_{n,g} u_g) \right) \\
&= \sum_{g,h\in\mathbb{F}_2} \tau\left(E_{q\mathcal{N}q}(u_g^* u_1^{t*} u_2^t y^* u_2^{t*} u_1^t u_h)\, E_{q\mathcal{N}q}(u_h^* u_1^{t*} u_2^t (b_{n,h}^* y b_{n,g}) u_2^{t*} u_1^t u_g) \right).
\end{aligned}$$

As in the proof of Theorem III.3.4, for $i \in \{1, 2\}$, put $G_1 = \langle a\rangle$ and $G_2 = \langle b\rangle$ so that $u_1 = u_a$ and $u_2 = u_b$. Denote by $(\beta_i(g))_{g\in G_i}$ the Fourier coefficients of u_i^t. For $g, h \in \mathbb{F}_2$, define the isometry $W_{g,h} : \mathrm{L}^2(\mathcal{M}_2) \ominus \mathrm{L}^2(\mathcal{B}) \to \mathrm{L}^2(\widetilde{\mathcal{M}})$ by $W_{g,h}(x) = u_g x u_h^*$ for $x \in \mathcal{M}_2 \ominus \mathcal{B}$ such that $\mathrm{Tr}_{\mathcal{M}}(x^* x) < \infty$. Thanks to Lemma III.3.2, the isometries $W_{g,h}$ have pairwise orthogonal ranges when (g, h) are pairwise distinct. For all $z \in q\mathcal{M}_2 q \ominus q\mathcal{B}q$ and all $g, h \in \mathbb{F}_2$, using calculation $(III.2)$, we obtain

$$\begin{aligned} E_{q\mathcal{N}q}(u_h^* u_1^{t*} u_2^t z u_2^{t*} u_1^t u_g) &= \sum_{r,r' \in G_1, s,s' \in G_2} \beta_1(r)\beta_2(s)\beta_2(s')\beta_1(r')\, E_{q\mathcal{N}q}(W_{h^{-1}r^{-1}s, g^{-1}r'^{-1}s'}(z)) \\ &= \sum_{\substack{r,r' \in G_1, s,s' \in G_2 \\ h^{-1}r^{-1}ss'^{-1}r'g=1}} \beta_1(r)\beta_2(s)\beta_2(s')\beta_1(r')\, W_{h^{-1}r^{-1}s, g^{-1}r'^{-1}s'}(z). \end{aligned}$$

Using the facts that $G_1 \cap G_2 = \{e\}$ and that the isometries $W_{g',h'}$ have pairwise orthogonal ranges when (g', h') are pairwise distinct, we get

$$\begin{aligned} &\tau\left(E_{q\mathcal{N}q}(u_g^* u_1^{t*} u_2^t y^* u_2^{t*} u_1^t u_h)\, E_{q\mathcal{N}q}(u_h^* u_1^{t*} u_2^t (b_{n,h}^* y b_{n,g}) u_2^{t*} u_1^t u_g)\right) \\ &= \sum_{\substack{r,r' \in G, s,s' \in G_2 \\ rss'r'=hg^{-1}}} \beta_1(r^{-1})^2 \beta_2(s)^2 \beta_2(s'^{-1})^2 \beta_1(r')^2\, \tau(y^* b_{n,h}^* y b_{n,g}). \end{aligned}$$

For $i \in \{1, 2\}$, define $\mu_i \in \mathrm{Prob}(\mathbb{F}_2)$ by $\mu_i(g) = \beta_i(g)^2$ if $g \in G_i$ and $\mu_i(g) = 0$ otherwise. Likewise, define $\check{\mu}_i \in \mathrm{Prob}(\mathbb{F}_2)$ by $\check{\mu}_i(g) = \mu_i(g^{-1})$ for all $g \in \mathbb{F}_2$. Put $\mu = \check{\mu}_1 * \mu_2 * \check{\mu}_2 * \mu_1$. Since $y \in q\mathcal{M}_2 q \ominus q\mathcal{B}q$ and $x_{n,g} \in q\mathcal{M}_1 q$, we obtain that

$$\begin{aligned} \tau\left(E_{q\mathcal{N}q}(u_g^* u_1^{t*} u_2^t y^* u_2^{t*} u_1^t u_h)\, E_{q\mathcal{N}q}(u_h^* u_1^{t*} u_2^t (b_{n,h}^* y b_{n,g}) u_2^{t*} u_1^t u_g)\right) &= \mu(hg^{-1})\, \tau(y^* b_{n,h}^* y b_{n,g}) \\ &= \mu(hg^{-1})\, \tau(y^* x_{n,h}^* y x_{n,g}). \end{aligned}$$

Summing over all $g, h \in \mathbb{F}_2$ and using Cauchy-Schwarz inequality, we get

$$\begin{aligned} |\langle \alpha_t(y)\eta_n \mid \eta_n \alpha_t(y)\rangle_{\mathrm{Tr}}| &= \left| \sum_{g,h \in \mathbb{F}_2} \mu(hg^{-1})\tau(y^* x_{n,h}^* y x_{n,g}) \right| \\ &= \left| \sum_{g,h \in \mathbb{F}_2} \mu(g)\tau(y^* x_{n,h}^* y x_{n,g^{-1}h}) \right| \\ &\leq \sum_{g,h \in \mathbb{F}_2} \mu(g) \|x_{n,h} y\|_2\, \|y x_{n,g^{-1}h}\|_2 \\ &\leq \sum_{g \in \mathbb{F}_2} \mu(g) \langle \zeta_n \mid \lambda_g(\zeta_n')\rangle_{\ell^2(\mathbb{F}_2)}, \end{aligned}$$

where $\zeta_n = \sum_{h \in \mathbb{F}_2} \|x_{n,h} y\|_2\, \delta_h$ and $\zeta_n' = \sum_{h \in \mathbb{F}_2} \|y x_{n,h}\|_2\, \delta_h$. Since we moreover have $u_1^{t*} \eta_n u_1^t = \sum_{g \in \mathbb{F}_2} u_g e_{q\mathcal{N}q} u_g^* x_{n,g}$, we get

$$\|u_1^{t*} \eta_n u_1^t y\|_{2,\mathrm{Tr}}^2 = \sum_{g \in \mathbb{F}_2} \|u_g e_{q\mathcal{N}q} u_g^* x_{n,g} y\|_{2,\mathrm{Tr}}^2 = \sum_{g \in \mathbb{F}_2} \|x_{n,g} y\|_2^2 = \|\zeta_n\|_{\ell^2(\mathbb{F}_2)}^2.$$

Likewise we have $\|\zeta_n'\|_{\ell^2(\mathbb{F}_2)} = \|y u_1^{t*} \eta_n u_1^t\|_{2,\mathrm{Tr}}$.

Denote by $T : \ell^2(\mathbb{F}_2) \to \ell^2(\mathbb{F}_2)$ the Markov operator defined by $T = \sum_{g \in \mathbb{F}_2} \mu(g)\lambda_g$. Since the support of μ generates $\mathbb{F}_2$ and $\mu(e) > 0$ (see the proof of [Io(12)a, Lemma 3.4, Claim]), Kesten's criterion for amenability [Ke59] yields $\|T\|_\infty < 1$. This gives

$$\begin{aligned}|\langle\alpha_t(y)\eta_n \,|\, \eta_n\alpha_t(y)\rangle_{\mathrm{Tr}}| &\le \langle\zeta_n \,|\, T\zeta_n'\rangle_{\ell^2(\mathbb{F}_2)}\\ &\le \|T\|_\infty \, \|\zeta_n\|_{\ell^2(\mathbb{F}_2)} \, \|\zeta_n'\|_{\ell^2(\mathbb{F}_2)}\\ &= \|T\|_\infty \, \|u_1^{t*}\eta_n u_1^t y\|_{2,\mathrm{Tr}} \, \|y u_1^{t*}\eta_n u_1^t\|_{2,\mathrm{Tr}}\\ &= \|T\|_\infty \, \|\eta_n u_1^t y\|_{2,\mathrm{Tr}} \, \|y u_1^{t*}\eta_n\|_{2,\mathrm{Tr}}.\end{aligned}$$

Since $\eta_n = \eta_n^*$, we obtain

$$\|\eta_n u_1^t y\|_{2,\mathrm{Tr}} \, \|y u_1^{t*}\eta_n\|_{2,\mathrm{Tr}} \to \|u_1^t y\|_2 \, \|y u_1^{t*}\|_2 = \|y\|_2^2 = 1,$$

hence $\limsup_n |\langle\alpha_t(y)\eta_n \,|\, \eta_n\alpha_t(y)\rangle_{\mathrm{Tr}}| \le \|T\|_\infty < 1$. This however contradicts the fact that

$$|\langle\alpha_t(y)\eta_n \,|\, \eta_n\alpha_t(y)\rangle_{\mathrm{Tr}}| \to 1$$

and hence our assumption that $q\mathcal{M}_1 q$ had no amenable direct summand. Thus for all $i \in \{1,2\}$, $q\mathcal{M}_i q$ has an amenable direct summand and so does $\mathcal{M}_i$. This finishes the proof of Theorem III.4.1. □

A combination of the proof of the above Theorem III.4.1 and the one of [Io(12)a, Theorem 4.1] shows that "or" can be replaced with "and" in Ioana's result [Io(12)a, Theorem 4.1].

Theorem III.4.3. *Let $M = M_1 *_B M_2$ be a tracial amalgamated free product von Neumann algebra. Assume that $M_1 \neq B \neq M_2$. Put $\widetilde{M} = M *_B (B \mathbin{\overline{\otimes}} \mathrm{L}(\mathbb{F}_2)) = N \rtimes \mathbb{F}_2$ where $N = \bigvee\{u_g M u_g^* : g \in \mathrm{L}(\mathbb{F}_2)\}$. Let $t \in (0,1)$ such that $\alpha_t(M)$ is amenable relative to N.*

Then for all $i \in \{1,2\}$, there exists a nonzero projection $z_i \in \mathcal{Z}(M_i)$ such that $M_i z_i$ is amenable relative to B inside M.

III.5 Proofs of Theorems III.A and III.B

III.5.1 A general intermediate result

Theorems III.A and III.B will be derived from the following very general result regarding Cartan subalgebras inside semifinite amalgamated free product von Neumann algebras.

Theorem III.5.1. *Let $\mathcal{M} = \mathcal{M}_1 *_\mathcal{B} \mathcal{M}_2$ be a semifinite amalgamated free product von Neumann algebra with semifinite faithful normal trace* Tr. *Assume that $\mathcal{B}$ is amenable, $\mathcal{M}_1$ has no amenable direct summand and for all nonzero projections $e \in \mathcal{B}$, we have $e\mathcal{B}e \neq e\mathcal{M}_2 e$.*

Let $p \in \mathrm{Proj_f}(\mathcal{B})$ and $\mathcal{A} \subset p\mathcal{M}p$ any regular amenable von Neumann subalgebra. Then there exists $q \in \mathrm{Proj_f}(\mathcal{B})$ such that $\mathcal{A} \preceq_\mathcal{M} q\mathcal{B}q$.

Proof. Put $\widetilde{\mathcal{M}} = \mathcal{M} *_\mathcal{B} (\mathcal{B} \mathbin{\overline{\otimes}} \mathrm{L}(\mathbb{F}_2))$ and regard $p\widetilde{\mathcal{M}}p$ as the tracial crossed product von Neumann algebra $p\widetilde{\mathcal{M}}p = p\mathcal{N}p \rtimes \mathbb{F}_2$ with $\mathcal{N} = \bigvee\{u_g \mathcal{M} u_g : g \in \mathbb{F}_2\}$. We denote by (α_t) the malleable deformation from Section III.3.1. Applying Popa-Vaes's dichotomy result [PV(12), Theorem 1.6] to the inclusion $\alpha_t(\mathcal{A}) \subset p\widetilde{\mathcal{M}}p$ for $t \in (0,1)$, we get that at least one of the following holds true:

1. Either $\alpha_t(\mathcal{A}) \preceq_{p\widetilde{\mathcal{M}}p} p\mathcal{N}p$.

2. Or $\alpha_t(p\mathcal{M}p)$ is amenable relative to $p\mathcal{N}p$ inside $p\widetilde{\mathcal{M}}p$.

Since $\mathcal{M}_1$ has no amenable direct summand, case (2) cannot hold by Theorem III.4.1. It remains to show that case (1) leads to the conclusion of the theorem.

In case (1), using Lemma III.2.3 and Theorem III.3.4, we get that either there exists $q \in \mathrm{Proj_f}(\mathcal{B})$ such that $\mathcal{A} \preceq_{\mathcal{M}} q\mathcal{B}q$ or there exist $i \in \{1,2\}$ and $q_i \in \mathrm{Proj_f}(\mathcal{M}_i)$ such that $p\mathcal{M}p \preceq_{\mathcal{M}} q_i\mathcal{M}_iq_i$. Since the latter case is impossible by Proposition III.2.6, we get $\mathcal{A} \preceq_{\mathcal{M}} q\mathcal{B}q$ for some $q \in \mathrm{Proj_f}(\mathcal{B})$. □

III.5.2 Proof of Theorem III.A

We first need to prove the following well-known result.

Lemma III.5.2. *Let M be any von Neumann algebra such that $M \neq \mathbb{C}$ and φ any faithful normal state on M. Realize the continuous core $\mathrm{c}(M) = M \rtimes_\varphi \mathbb{R}$. Then for every nonzero projection $p \in \mathrm{L}(\mathbb{R})$, we have $\mathrm{L}(\mathbb{R})p \neq p\mathrm{c}(M)p$.*

Proof. There are two cases to consider.

Case (1): assume that $M^\varphi \neq \mathbb{C}$. Choose $r \in M^\varphi$ a projection such that $r \neq 0, 1$. Observe that $x = \varphi(1-r)\, r - \varphi(r)\,(1-r) \in M^\varphi$ is invertible and $\varphi(x) = 0$. Then for every nonzero projection $p \in \mathrm{L}(\mathbb{R})$, we have $xp \neq 0$ and $E_{\mathrm{L}(\mathbb{R})p}(xp) = \varphi(x)p = 0$. This proves that $\mathrm{L}(\mathbb{R})p \neq pMp$.

Case (2): assume that $M^\varphi = \mathbb{C}$. Since $\mathcal{Z}(M) \subset \mathcal{Z}(M^\varphi)$, it follows that M is a factor. If M is of type III, it follows from Connes's classification of type III factors [Co73] that M is necessarily of type III_1. In that case, $\mathrm{c}(M)$ is a type II_∞ factor and thus $\mathrm{L}(\mathbb{R})p \neq p\mathrm{c}(M)p$ for every nonzero projection $p \in \mathrm{L}(\mathbb{R})$. If M is a semifinite factor with semifinite faithful normal trace Tr, there exists $b \in \mathrm{L}^1(M, \mathrm{Tr})_+$ such that $\varphi = \mathrm{Tr}(b\,\cdot)$ and $\|b\|_{1,\mathrm{Tr}} = 1$. Let $q \in M$ be a nonzero spectral projection of b. Since

$$\varphi(qx) = \mathrm{Tr}(bqx) = \mathrm{Tr}(qbx) = \mathrm{Tr}(bxq) = \varphi(xq)$$

for all $x \in M$, we get $q \in M^\varphi$ and so $q = 1$. This shows that $b = 1$ and $\mathrm{Tr} = \varphi$ is a finite trace on M. Hence $M = M^\varphi = \mathbb{C}$, which is a contradiction. □

Proof of Theorem III.A. By [Ue11, Theorem 4.1], we know that there exists a nonzero projection $z \in \mathcal{Z}(M)$ such that Mz is a full factor and $M(1-z)$ is a purely atomic von Neumann algebra. In particular, M is not amenable.

In the case when both M_1 and M_2 are amenable, [HR11, Theorem 5.5] implies that M has no Cartan subalgebra. It remains to consider the case when M_1 or M_2 is not amenable. Without loss of generality, we may assume that M_1 is not amenable.

By contradiction, assume that M has a Cartan subalgebra. Hence, Mz also has a Cartan subalgebra. Let $p \in \mathcal{Z}(M_1)$ be the largest nonzero projection such that M_1p has no amenable direct summand. Since $M(1-z)$ is purely atomic, we necessarily have $p \leq z$.

By [Ue11, Lemma 2.2], we have

$$(pMp, \frac{1}{\varphi(p)}\varphi(p\cdot p)) = (M_1p, \frac{1}{\varphi_1(p)}\varphi_1(\cdot p)) * (pNp, \frac{1}{\varphi(p)}\varphi(p\cdot p))$$

with $N = (\mathbb{C}p \oplus M_1(1-p)) \vee M_2$. Observe that $pNp \neq \mathbb{C}p$. Indeed let $q \in M_2$ be a projection such that $q \neq 0, 1$. Then $pqp = \varphi_2(q)p + p(q - \varphi_2(q))p \in pNp \setminus \mathbb{C}p$. Since Mz is a factor and $p \leq z$, it follows that pMp has a Cartan subalgebra by [Po06a, Lemma 3.5].

From the previous discussion, it follows that we may assume that M_1 has no amenable direct summand, $M_2 \neq \mathbb{C}$ and M has a Cartan subalgebra $A \subset M$. Using Notation III.2.9, denote by $\mathrm{c}(A) \subset \mathrm{c}(M)$ the Cartan subalgebra in the continuous core $\mathrm{c}(M) = \mathrm{c}(M_1) *_{\mathrm{L}(\mathbb{R})} \mathrm{c}(M_2)$. Let $q \in \mathrm{Proj_f}(\mathrm{L}(\mathbb{R}))$. Since $\mathrm{c}(A) \subset \mathrm{c}(M)$ is maximal abelian and $\mathrm{Tr}\,|\mathrm{c}(A)$ is semifinite, [HV13, Lemma 2.1] shows that there exists a nonzero finite trace projection $p \in \mathrm{c}(A)$ and a partial isometry $v \in \mathrm{c}(M)$ such that $p = v^*v$ and $q = vv^*$. Observe that $v\mathrm{c}(A)v^* \subset q\mathrm{c}(M)q$ is still a Cartan subalgebra by [Po06a, Lemma 3.5].

By Lemma III.2.3, Proposition III.2.8, Theorem III.5.1 and Lemma III.5.2, there exists $q' \in \mathrm{Proj_f}(\mathrm{L}(\mathbb{R}))$ such that $v\mathrm{c}(A)v^* \preceq_{\mathrm{c}(M)} \mathrm{L}(\mathbb{R})q'$. Then Proposition III.2.10 implies that $A \preceq_M \mathbb{C}$. This contradicts the fact that A is diffuse and finishes the proof of Theorem III.A. □

III.5.3 Proof of Theorem III.B

Proof of Theorem III.B. Let $A \subset M$ be a Cartan subalgebra. Since $A, B \subset M$ are both tracial von Neumann subalgebras of M with expectation, we use Notation III.2.9. Let $q \in \mathrm{Proj_f}(\mathcal{Z}(\mathrm{c}(B)))$. By [HV13, Lemma 2.1], there exists $p \in \mathrm{Proj_f}(\mathrm{c}(A))$ and a partial isometry $v \in \mathrm{c}(M)$ such that $p = v^*v$ and $q = vv^*$. Observe that $v\mathrm{c}(A)v^* \subset q\mathrm{c}(M)q$ is still a Cartan subalgebra by [Po06a, Lemma 3.5].

Using the assumptions, by Lemma III.2.3, Proposition III.2.8, [HV13, Proposition 5.5] and Theorem III.5.1, there exists $q' \in \mathrm{Proj_f}(\mathcal{Z}(\mathrm{c}(B)))$ such that $v\mathrm{c}(A)v^* \preceq_{\mathrm{c}(M)} \mathrm{c}(B)q'$. Then Proposition III.2.10 implies that $A \preceq_M B$. □

III.6 Proof of Theorem III.C

Let $\mathcal{R}$ be any countable nonsingular equivalence relation on a standard measure space (X, μ). Following [FM77], denote by m the measure on $\mathcal{R}$ given by

$$m(\mathcal{W}) = \int_X |\{y \in X : (x, y) \in \mathcal{W}\}| \, \mathrm{d}\mu(x)$$

for all measurable subsets $\mathcal{W} \subset \mathcal{R}$. We denote by $[\mathcal{R}]$ the full group of $\mathcal{R}$, $M = \mathrm{L}(\mathcal{R})$ the von Neumann algebra of $\mathcal{R}$ and identify $\mathrm{L}^2(M) = \mathrm{L}^2(\mathcal{R}, m)$. For all $\psi \in [\mathcal{R}]$, define $u(\psi) \in \mathcal{U}(M)$ whose action on $\mathrm{L}^2(\mathcal{R}, m)$ is given by

$$(u(\psi)\xi)(x, y) = \left(\frac{\mathrm{d}(\mu \circ \psi^{-1})}{\mathrm{d}\mu}(x)\right)^{1/2} \xi(\psi^{-1}(x), y).$$

We view $\mathrm{L}^\infty(\mathcal{R})$ as acting on $\mathrm{L}^2(\mathcal{R}, m)$ by multiplication operators. Note that the unitaries $u(\psi) \in \mathcal{U}(M)$ for $\psi \in [\mathcal{R}]$ normalize $\mathrm{L}^\infty(\mathcal{R})$ and that $\mathrm{L}^\infty(X) \subset \mathrm{L}^\infty(\mathcal{R})$, by identifying a function $F \in \mathrm{L}^\infty(X)$ with the function on $\mathcal{R}$ given by $(x, y) \mapsto F(x)$.

Recall from [CFW81, Definition 5] that $\mathcal{R}$ is *amenable* if there exists a norm one projection $\Phi : \mathrm{L}^\infty(\mathcal{R}) \to \mathrm{L}^\infty(X)$ satisfying

$$\Phi(u(\psi)Fu(\psi)^*) = u(\psi)\Phi(F)u(\psi)^*, \forall \psi \in [\mathcal{R}].$$

By [CFW81, Theorem 10], a countable nonsingular equivalence relation $\mathcal{R}$ is amenable if and only if it is hyperfinite. We will say that a countable nonsingular equivalence relation $\mathcal{R}$ is *nowhere amenable* if for every measurable subset $\mathcal{U} \subset X$ such that $\mu(\mathcal{U}) > 0$, the equivalence relation $\mathcal{R}|\mathcal{U} = \mathcal{R} \cap (\mathcal{U} \times \mathcal{U})$ is nonamenable.

Recall the following definition due to Gaboriau [Ga00, Definition IV.6].

Definition III.6.1. Let $\mathcal{R}$ be a countable nonsingular equivalence relation on a standard measure space (X, μ) and $\mathcal{R}_1, \mathcal{R}_2 \subset \mathcal{R}$ subequivalence relations. We say that $\mathcal{R}$ *splits as the free product* $\mathcal{R} = \mathcal{R}_1 * \mathcal{R}_2$ if

- $\mathcal{R}$ is generated by $\mathcal{R}_1$ and $\mathcal{R}_2$;
- For every $p \in \mathbb{N}_{>0}$ and almost every $2p$-tuple $(x_j)_{j\in\mathbb{Z}/2p\mathbb{Z}}$ in X such that $(x_{2i-1}, x_{2i}) \in \mathcal{R}_1$ and $(x_{2i}, x_{2i+1}) \in \mathcal{R}_2$ for all $i \in \mathbb{Z}/p\mathbb{Z}$, there exists $j \in \mathbb{Z}/2p\mathbb{Z}$ such that $x_j = x_{j+1}$.

We have the following well-known fact:

Proposition III.6.2. *Let $\mathcal{R}$ be a countable nonsingular equivalence relation on a standard measure space (X, μ) and $\mathcal{R}_1, \mathcal{R}_2 \subset \mathcal{R}$ subequivalence relations. Let $B = \mathrm{L}^\infty(X)$, $M_1 = \mathrm{L}(\mathcal{R}_1)$, $M_2 = \mathrm{L}(\mathcal{R}_2)$, $M = \mathrm{L}(\mathcal{R})$ and denote by $E_1 : M_1 \to B$, $E_2 : M_2 \to B$, $E : M \to B$ the canonical faithful normal conditional expectations. The following conditions are equivalent:*

1. *$\mathcal{R}$ splits as the free product $\mathcal{R} = \mathcal{R}_1 * \mathcal{R}_2$.*
2. *$(M, E) = (M_1, E_1) *_B (M_2, E_2)$*

We start by proving the following intermediate result in the framework of type II_1 equivalence relations.

Theorem III.6.3. *Let $\mathcal{R}$ be a countable (not necessarily ergodic) probability measure preserving equivalence relation on a standard probability space (X, μ) which splits as a free product $\mathcal{R} = \mathcal{R}_1 * \mathcal{R}_2$ where $\mathcal{R}_i$ is a countable type II_1 subequivalence relation for all $i \in \{1, 2\}$.*

Let $A \subset \mathrm{L}(\mathcal{R})$ be a Cartan subalgebra. Then $A \preceq_{\mathrm{L}(\mathcal{R})} \mathrm{L}^\infty(X)$.

Proof. Let $B = \mathrm{L}^\infty(X)$, $M_1 = \mathrm{L}(\mathcal{R}_1)$, $M_2 = \mathrm{L}(\mathcal{R}_2)$ and $M = \mathrm{L}(\mathcal{R})$ so that $M = M_1 *_B M_2$. Let $A \subset M$ be a Cartan subalgebra.

Assume first that both $\mathcal{R}_1$ and $\mathcal{R}_2$ are amenable and thus hyperfinite by [CFW81]. Since both $\mathcal{R}_1$ and $\mathcal{R}_2$ are moreover of type II_1, they are necessarily generated by a free pmp action of $\mathbb{Z}$. Hence $\mathcal{R} = \mathcal{R}_1 * \mathcal{R}_2$ is generated by a free pmp action of $\mathbb{F}_2$ and so $M \cong B \rtimes \mathbb{F}_2$. Then [PV(12), Theorem 1.6] shows that $A \preceq_M B$.

Next assume that $\mathcal{R}_1$ or $\mathcal{R}_2$ is nonamenable. Without loss of generality, we may assume that $\mathcal{R}_1$ is nonamenable. Choose a measurable subset $\mathcal{U} \subset X$ such that $\mu(\mathcal{U}) > 0$ and $\mathcal{R}_1|\mathcal{U}$ is nowhere amenable. Denote by $\mathcal{V} \subset X$ the $\mathcal{R}$-saturated measurable subset of $\mathcal{U}$ in X. Since $\mathcal{R}|\mathcal{V} = (\mathcal{R}_1|\mathcal{V}) * (\mathcal{R}_2|\mathcal{V})$, we may assume that $\mu(\mathcal{V}) = 1$.

Since $\mathcal{U}$ is a complete section for $\mathcal{R}$, it follows from [Al10, Théorème 44] that we can write $\mathcal{R}|\mathcal{U} = \mathcal{S}_1 * \mathcal{S}_2$ where $\mathcal{S}_1 = \mathcal{R}_1|\mathcal{U}$ and $\mathcal{S}_2$ is a type II_1 subequivalence relation of $\mathcal{R}|\mathcal{U}$ which contains $\mathcal{R}_2|\mathcal{U}$.

Write $q = \mathbf{1}_{\mathcal{U}} \in B$. By [BO08, Corollary F.8], choose a projection $p \in A$ and a partial isometry $v \in M$ such that $v^*v = p$ and $vv^* = q$. Then $vAv^* \subset qMq$ is a Cartan subalgebra by [Po06a, Lemma 3.5]. We can thus apply Theorem III.5.1 to $\mathcal{M} = \mathrm{L}(\mathcal{S}_1) *_{\mathrm{L}^\infty(\mathcal{U})} \mathrm{L}(\mathcal{S}_2)$, $\mathcal{A} = vAv^*$ and $p = 1$. Then we obtain that $vAv^* \preceq_{qMq} Bq$, hence $A \preceq_M B$. □

Proof of Theorem III.C. Write $B = \mathrm{L}^\infty(X)$, $M_1 = \mathrm{L}(\mathcal{R}_1)$, $M_2 = \mathrm{L}(\mathcal{R}_2)$ and $M = \mathrm{L}(\mathcal{R})$ so that $M = M_1 *_B M_2$. Define on the standard infinite measure space $(X \times \mathbb{R}, m)$ the countable infinite measure preserving equivalence relations $\mathrm{c}(\mathcal{R}_1)$, $\mathrm{c}(\mathcal{R}_2)$ and $\mathrm{c}(\mathcal{R})$ which are the Maharam extensions [Ma64] of the countable nonsingular equivalence relations $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}$ respectively. Observe that both $\mathrm{c}(\mathcal{R}_1)$ and $\mathrm{c}(\mathcal{R}_2)$ are of type II and $\mathrm{c}(\mathcal{R}) = \mathrm{c}(\mathcal{R}_1) * \mathrm{c}(\mathcal{R}_2)$.

If we moreover write $\mathrm{c}(B) = \mathrm{L}^\infty(X \times \mathbb{R})$, we canonically have

$$\mathrm{c}(M_1) = \mathrm{L}(\mathrm{c}(\mathcal{R}_1)),\ \mathrm{c}(M_2) = \mathrm{L}(\mathrm{c}(\mathcal{R}_2)),\ \mathrm{c}(M) = \mathrm{L}(\mathrm{c}(\mathcal{R}))\ \text{ and }\ \mathrm{c}(M) = \mathrm{c}(M_1) *_{\mathrm{c}(B)} \mathrm{c}(M_2).$$

Let $A \subset M$ be a Cartan subalgebra. Using Notation III.2.9, we obtain that $\mathrm{c}(A) \subset \mathrm{c}(M)$ is a Cartan subalgebra. Let $q \in \mathrm{Proj}_{\mathrm{f}}(\mathrm{c}(B))$ such that $\mathrm{Tr}(q) = 1$. Up to cutting down by the central support of q in $\mathrm{c}(M)$, we may assume that q has central support equal to 1 in $\mathrm{c}(M)$. By [HV13, Lemma 2.1], there exists $p \in \mathrm{Proj}_{\mathrm{f}}(\mathrm{c}(A))$ and a partial isometry $v \in \mathrm{c}(M)$ such that $p = v^*v$ and $q = vv^*$. Observe that $v\mathrm{c}(A)v^* \subset q\mathrm{c}(M)q$ is still a Cartan subalgebra by [Po06a, Lemma 3.5]. In order to show that A and B are unitarily conjugate inside M, using Theorem III.2.1 and Proposition III.2.10, it suffices to show that $v\mathrm{c}(A)v^* \preceq_{\mathrm{c}(M)} \mathrm{c}(B)q$.

Let $\mathcal{U} \subset X \times \mathbb{R}$ be a measurable subset such that $\mathbf{1}_{\mathcal{U}} = q$. Since $\mathbf{1}_{\mathcal{U}}$ has central support equal to 1 in $\mathrm{c}(M)$, $\mathcal{U}$ is a complete section for $\mathrm{c}(\mathcal{R})$. By [Al10, Théorème 44], we can write $\mathrm{c}(\mathcal{R})|\mathcal{U} = \mathcal{S}_1 * \mathcal{S}_2$ where $\mathcal{S}_1 = \mathrm{c}(\mathcal{R}_1)|\mathcal{U}$ and $\mathcal{S}_2$ is a subequivalence relation of $\mathrm{c}(\mathcal{R})|\mathcal{U}$ which contains $\mathrm{c}(\mathcal{R}_2)|\mathcal{U}$. In particular, both $\mathcal{S}_1$ and $\mathcal{S}_2$ are type II_1 equivalence relations on the standard probability space $(\mathcal{U}, m|\mathcal{U})$.

Let $\mathcal{A} = v\mathrm{c}(A)v^*$ and $\mathcal{B} = \mathrm{L}^\infty(\mathcal{U})$. Observe that $q\mathrm{c}(M)q = \mathrm{L}(\mathrm{c}(\mathcal{R})|\mathcal{U}) = \mathrm{L}(\mathcal{S}_1 * \mathcal{S}_2)$ and $\mathcal{A}$ is a Cartan subalgebra in $\mathrm{L}(\mathcal{S}_1 * \mathcal{S}_2)$. Then Theorem III.6.3 implies that $\mathcal{A} \preceq_{\mathrm{L}(\mathcal{S}_1 * \mathcal{S}_2)} \mathrm{L}^\infty(\mathcal{U})$, that is, $v\mathrm{c}(A)v^* \preceq_{\mathrm{c}(M)} \mathrm{c}(B)q$. This finishes the proof of Theorem III.C. □

III.7 Proof of Theorem III.D

We start by proving Theorem III.D in the infinite measure preserving case. More precisely, we deduce the following result from its finite measure preserving counterpart proven in [Io(12)a, Theorem 1.1].

Theorem III.7.1. *Let $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ be an amalgamated product group such that Σ is finite and for all $i \in \{1, 2\}$, Γ_i is infinite. Let $(\mathcal{B}, \mathrm{Tr})$ be a type* I *von Neumann algebra endowed with a semifinite faithful normal trace. Let $\Gamma \curvearrowright (\mathcal{B}, \mathrm{Tr})$ be a trace preserving action such that for all $i \in \{1, 2\}$, the crossed product von Neumann algebra $\mathcal{B} \rtimes \Gamma_i$ is of type* II. *Put $\mathcal{M} = \mathcal{B} \rtimes \Gamma$. Let $p \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$ and $\mathcal{A} \subset p\mathcal{M}p$ any regular amenable von Neumann subalgebra.*

Then for every nonzero projection $e \in \mathcal{A}' \cap p\mathcal{M}p$, we have $\mathcal{A}e \preceq_{p\mathcal{M}p} p\mathcal{B}p$.

Proof. For every subset $\mathcal{F} \subset \Gamma$, denote by $P_{\mathcal{F}}$ the orthogonal projection from $\mathrm{L}^2(\mathcal{M}, \mathrm{Tr})$ onto the closed linear span of $\{xu_g : x \in \mathcal{B} \cap \mathrm{L}^2(\mathcal{B}, \mathrm{Tr}), g \in \mathcal{F}\}$. Since $\mathcal{N}_{p\mathcal{M}p}(\mathcal{A})'' = p\mathcal{M}p$, Proposition III.2.4 (see also [HV13, Lemma 2.7]) provides a central projection $z \in \mathcal{Z}(p\mathcal{M}p)$ and a net of unitaries $w_k \in \mathcal{U}(\mathcal{A}z)$ such that:

- $\lim_k \|P_{\mathcal{F}}(w_k)\|_{2,\mathrm{Tr}} = 0$ for all finite subset $\mathcal{F} \subset \Gamma$.
- For every $\varepsilon > 0$, there exists a finite subset $\mathcal{F} \subset \Gamma$ such that $\|a - P_{\mathcal{F}}(a)\|_{2,\mathrm{Tr}} \leq \varepsilon$ for all $a \in \mathrm{Ball}(\mathcal{A}(p - z))$.

We prove by contradiction that $z = 0$. So, assume that $z \neq 0$. Recall that $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$. Hence the subgroup $\Sigma_0 = \bigcap_{g \in \Gamma} g\Sigma g^{-1} < \Sigma$ is finite and normal in Γ. Define the quotient homomorphism $\rho : \Gamma \to \Gamma/\Sigma_0$ and put $\Lambda = \Gamma/\Sigma_0$, $\Lambda_i = \Gamma_i/\Sigma_0$ for $i \in \{1, 2\}$, $\Upsilon = \Sigma/\Sigma_0$ so that $\Lambda = \Lambda_1 *_\Upsilon \Lambda_2$. We get that $\bigcap_{s \in \Lambda} s\Upsilon s^{-1} = \{e\}$, hence $\mathrm{L}(\Lambda)$ is a II_1 factor which does not have property Gamma by [Io(12)a, Corollary 6.2].

Define the unitary $W \in \mathcal{U}(\mathrm{L}^2(\mathcal{B}, \mathrm{Tr}) \otimes \ell^2(\Gamma) \otimes \ell^2(\Lambda))$ by

$$W(\xi \otimes \delta_g \otimes \delta_s) = \xi \otimes \delta_g \otimes \delta_{\rho(g^{-1})s}, \forall \xi \in \mathrm{L}^2(\mathcal{B}, \mathrm{Tr}), \forall g \in \Gamma, \forall s \in \Lambda.$$

Next, define the *dual coaction* $\Delta_\rho : \mathcal{M} \to \mathcal{M} \mathbin{\overline{\otimes}} \mathrm{L}(\Lambda)$ by $\Delta_\rho(x) = W^*(x \otimes 1)W$ for all $x \in \mathcal{M}$. Observe that Δ_ρ is a trace preserving $*$-embedding which satisfies $\Delta_\rho(bu_g) = bu_g \otimes v_{\rho(g)}$ for all $b \in \mathcal{B}$ and all $g \in \Gamma$.

For every subset $\mathcal{F} \subset \Gamma$, denote by $Q_{\rho(\mathcal{F})}$ the orthogonal projection from $\mathrm{L}^2(\mathrm{L}(\Lambda))$ onto the closed linear span of $\{v_{\rho(g)} : g \in \mathcal{F}\}$. Observe that $(1 \otimes Q_{\rho(\mathcal{F})})(\Delta_\rho(x)) = \Delta_\rho(P_{\Sigma_0\mathcal{F}}(x))$ for all $x \in \mathcal{M}$. Since Δ_ρ is $\|\cdot\|_{2,\mathrm{Tr}}$-preserving and since Σ_0 is finite, for any finite subset $\mathcal{F} \subset \Gamma$, we have

$$\lim_k \|(1 \otimes Q_{\rho(\mathcal{F})})(\Delta_\rho(w_k))\|_2 = \lim_k \|\Delta_\rho(P_{\Sigma_0\mathcal{F}}(w_k))\|_2 = 0.$$

Since $\Upsilon < \Lambda$ is a finite subgroup, this implies that $\Delta_\rho(\mathcal{A}z) \not\preceq_{\mathcal{M}\overline{\otimes}\mathrm{L}(\Lambda)} q\mathcal{M}q \mathbin{\overline{\otimes}} \mathrm{L}(\Upsilon)$ for all $q \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$.

Put $\widetilde{\Lambda} = \Lambda *_\Upsilon (\Upsilon \times \mathbb{F}_2) = \Lambda_1 *_\Upsilon \Lambda_2 *_\Upsilon (\Upsilon \times \mathbb{F}_2)$ and consider the malleable deformation (α_t) on $\mathrm{L}(\widetilde{\Lambda})$ from Section III.3.1. Define $N < \Lambda$ the normal subgroup generated by $\{t\Lambda t^{-1} : t \in \mathbb{F}_2\}$ so that $\mathrm{L}(\widetilde{\Lambda}) = \mathcal{N} \rtimes \mathbb{F}_2$ with $\mathcal{N} = \mathrm{L}(N)$. Applying Popa-Vaes's dichotomy result [PV(12), Theorem 1.6] to each of the inclusions

$$(\mathrm{id} \otimes \alpha_t)(\Delta_\rho(\mathcal{A}z)) \subset p\mathcal{M}p \mathbin{\overline{\otimes}} \mathrm{L}(\widetilde{\Lambda}) = (p\mathcal{M}p \mathbin{\overline{\otimes}} \mathcal{N}) \rtimes \mathbb{F}_2 \;\text{ with }\; t \in (0, 1),$$

we obtain that at least one of the following holds true:

1. Either there exists $t \in (0, 1)$ such that $(\mathrm{id} \otimes \alpha_t)(\Delta_\rho(\mathcal{A}z)) \preceq_{p\mathcal{M}p\overline{\otimes}\mathrm{L}(\widetilde{\Lambda})} p\mathcal{M}p \mathbin{\overline{\otimes}} \mathcal{N}$.
2. Or for all $t \in (0, 1)$, $(\mathrm{id} \otimes \alpha_t)(\Delta_\rho(p\mathcal{M}p))$ is amenable relative to $p\mathcal{M}p \mathbin{\overline{\otimes}} \mathcal{N}$ inside $p\mathcal{M}p \mathbin{\overline{\otimes}} \mathrm{L}(\widetilde{\Lambda})$.

We will prove below that each case leads to a contradiction.

In case (1), by [Io(12)a, Theorem 3.2] and since $\Delta_\rho(\mathcal{A}z) \not\preceq_{p\mathcal{M}p\overline{\otimes}\mathrm{L}(\Lambda)} p\mathcal{M}p\overline{\otimes}\mathrm{L}(\Upsilon)$ and $\mathcal{N}_{p\mathcal{M}pz}(\mathcal{A}z)'' = p\mathcal{M}pz$, there exists $i \in \{1, 2\}$ such that $\Delta_\rho(p\mathcal{M}pz) \preceq_{p\mathcal{M}p\overline{\otimes}\mathrm{L}(\Lambda)} p\mathcal{M}p \mathbin{\overline{\otimes}} \mathrm{L}(\Lambda_i)$. In order to get a contradiction, we will need the following.

Claim. Let $e \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{M})$, $\mathcal{Q} \subset e\mathcal{M}e$ any von Neumann subalgebra and $\mathcal{S}$ any nonempty collection of subgroups of Γ. If $\mathcal{Q} \not\preceq_{\mathcal{M}} q(\mathcal{B} \rtimes H)q$ for all $H \in \mathcal{S}$ and all $q \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$, then $\Delta_\rho(\mathcal{Q}) \not\preceq_{\mathcal{M}\overline{\otimes}\mathrm{L}(\Lambda)} q\mathcal{M}q \mathbin{\overline{\otimes}} \mathrm{L}(\rho(H))$ for all $H \in \mathcal{S}$ and all $q \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$.

Proof of the Claim. Since $\mathcal{Q} \npreceq_{\mathcal{M}} q(\mathcal{B} \rtimes H)q$ for all $H \in \mathcal{S}$ and all $q \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$, Proposition III.2.4 implies that there exists a net $v_k \in \mathcal{U}(\mathcal{Q})$ such that $\lim_k \|P_{\mathcal{F}}(v_k)\|_{2,\mathrm{Tr}} = 0$ for all subsets $\mathcal{F} \subset \Gamma$ which are small relative to $\mathcal{S}$. Observe that since Σ_0 is finite, $\Sigma_0\mathcal{F}$ is small relative to $\mathcal{S}$ for all subsets $\mathcal{F} \subset \Gamma$ which are small relative to $\mathcal{S}$. Moreover, $(1 \otimes Q_{\rho(\mathcal{F})})(\Delta_\rho(x)) = \Delta_\rho(P_{\Sigma_0\mathcal{F}}(x))$ for all $x \in \mathcal{Q}$ and all subsets $\mathcal{F} \subset \Gamma$ which are small relative to $\mathcal{S}$. Since Δ_ρ is $\|\cdot\|_{2,\mathrm{Tr}}$-preserving, for all subsets $\mathcal{F} \subset \Gamma$ which are small relative to $\mathcal{S}$, we have

$$\lim_k \|(1 \otimes Q_{\rho(\mathcal{F})})(\Delta_\rho(v_k))\|_{2,\mathrm{Tr}} = \lim_k \|\Delta_\rho(P_{\Sigma_0\mathcal{F}}(v_k))\|_{2,\mathrm{Tr}} = 0. \tag{III.5}$$

Denote by $\rho(\mathcal{S})$ the nonempty collection of subgroups $\rho(H) \subset \Lambda$ with $H \in \mathcal{S}$. Let $\mathcal{G} \subset \Lambda$ be any subset which is small relative to $\rho(\mathcal{S})$. Then there exist $n \geq 1$, $H_1, \dots, H_n \in \mathcal{S}$ and $s_1, t_1, \dots, s_n, t_n \in \Lambda$ such that $\mathcal{G} \subset \bigcup_{i=1}^n s_i\rho(H_i)t_i$. Choose $g_i, h_i \in \Gamma$ such that $\rho(g_i) = s_i$ and $\rho(h_i) = t_i$ and denote $\mathcal{F} = \bigcup_{i=1}^n g_iH_ih_i$. Then $\mathcal{G} \subset \rho(\mathcal{F})$. Therefore, $(III.5)$ implies that $\lim_k \|(1 \otimes Q_{\mathcal{G}})(\Delta_\rho(v_k))\|_{2,\mathrm{Tr}} = 0$ for all subsets $\mathcal{G} \subset \Lambda$ which are small relative to $\rho(\mathcal{S})$. Thus, Proposition III.2.4 implies that $\Delta_\rho(\mathcal{Q}) \npreceq_{\mathcal{M}\overline{\otimes}\mathrm{L}(\Lambda)} q\mathcal{M}q \mathbin{\overline{\otimes}} \mathrm{L}(\rho(H))$ for all $H \in \mathcal{S}$ and all $q \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$. ☐

We apply the Claim to $\mathcal{Q} = p\mathcal{M}pz$ and $\mathcal{S} = \{\Gamma_1, \Gamma_2\}$. In order to do that, we need to check that $p\mathcal{M}pz \npreceq_{q\mathcal{M}q} q(\mathcal{B} \rtimes \Gamma_i)q$ for all $i \in \{1,2\}$ and all $q \in \mathrm{Proj}_{\mathrm{f}}(\mathcal{B})$. Since $\mathcal{B} \rtimes \Sigma$ is a type I von Neumann algebra and $\mathcal{B} \rtimes \Gamma_i$ is a type II von Neumann algebra, Proposition III.2.6 yields the result. Therefore, by the Claim, we get that $\Delta_\rho(p\mathcal{M}pz) \npreceq_{p\mathcal{M}p\overline{\otimes}\mathrm{L}(\Lambda)} p\mathcal{M}p \mathbin{\overline{\otimes}} \mathrm{L}(\Lambda_i)$ for all $i \in \{1,2\}$. This is a contradiction.

In case (2), since $\mathrm{L}(\Lambda)$ does not have property Gamma, [Io(12)a, Theorem 5.2] shows that either there exists $i \in \{1,2\}$ such that $\mathrm{L}(\Lambda) \preceq_{\mathrm{L}(\Lambda)} \mathrm{L}(\Lambda_i)$ or $\mathrm{L}(\Lambda)$ is amenable. Both of these cases are easily seen to lead to a contradiction. This finishes the proof of Theorem III.7.1. ☐

Proof of Theorem III.D. Let now $\Gamma \curvearrowright (X,\mu)$ be any nonsingular free ergodic action on a standard measure space such that for all $i \in \{1,2\}$, the restricted action $\Gamma_i \curvearrowright (X,\mu)$ is recurrent. Let $B = \mathrm{L}^\infty(X)$ and put $M = B \rtimes \Gamma$. Assume that $A \subset M$ is another Cartan subalgebra.

Since $A, B \subset M$ are both tracial von Neumann subalgebras of M with expectation, we use Notation III.2.9. Define $\mathrm{c}(B) = \mathrm{L}^\infty(X \times \mathbb{R})$ and consider the Maharam extension $\Gamma \curvearrowright \mathrm{c}(B)$ of the action $\Gamma \curvearrowright B$ so that we canonically have $\mathrm{c}(M) = \mathrm{c}(B) \rtimes \Gamma$. Observe that for all $i \in \{1,2\}$, the action $\Gamma_i \curvearrowright \mathrm{c}(B)$ is still recurrent so that $\mathrm{c}(B) \rtimes \Gamma_i$ is a type II von Neumann algebra.

Let $p \in \mathrm{Proj}_{\mathrm{f}}(\mathrm{c}(A))$. By [HV13, Lemma 2.1], there exist $q \in \mathrm{Proj}_{\mathrm{f}}(\mathrm{c}(B))$ and a partial isometry $v \in \mathrm{c}(M)$ such that $p = v^*v$ and $q = vv^*$. Observe that $v\mathrm{c}(A)v^* \subset q\mathrm{c}(M)q$ is still a Cartan subalgebra by [Po06a, Lemma 3.5].

By Theorem III.7.1, we get $v\mathrm{c}(A)v^* \preceq_{q\mathrm{c}(M)q} \mathrm{c}(B)q$. By Proposition III.2.10, this implies that $A \preceq_M B$. Since M is a factor, by [HV13, Theorem 2.5], we get that there exists a unitary $u \in \mathcal{U}(M)$ such that $uAu^* = B$. This finishes the proof of Theorem III.D. ☐

III.8 AFP von Neumann algebras with many Cartan subalgebras

Connes and Jones exhibited in [CJ82] the first examples of II_1 factors M with at least two Cartan subalgebras which are not conjugate by an automorphism of M. More concrete examples were found by Ozawa and Popa in [OP10b].

Recently, Speelman and Vaes exhibited in [SV12] the first examples of group measure space II_1 factors $M = \mathrm{L}^\infty(Y) \rtimes \Lambda$ with uncountably many non stably conjugate Cartan subalgebras. Recall from [SV12] that two Cartan subalgebras A and B of a II_1 factor N are *stably conjugate* if there exists nonzero projections $p \in A$ and $q \in B$ and a surjective $*$-isomorphism $\alpha : pNp \to qNq$ such that $\alpha(Ap) = Bq$. Put $\mathcal{N} = N \mathbin{\overline{\otimes}} \mathbf{B}(\ell^2)$, $\mathcal{A} = A \mathbin{\overline{\otimes}} \ell^\infty$ and $\mathcal{B} = B \mathbin{\overline{\otimes}} \ell^\infty$. Observe that $\mathcal{A}$ and $\mathcal{B}$ are Cartan subalgebras in the type II_∞ factor $\mathcal{N}$. Moreover, we have that A and B are stably conjugate in N if and only if $\mathcal{A}$ and $\mathcal{B}$ are conjugate in $\mathcal{N}$.

Let $\Lambda \curvearrowright (Y, \nu)$ be a probability measure preserving free ergodic action as in the statement of [SV12, Theorem 2] so that the corresponding group measure space II_1 factor $N = \mathrm{L}^\infty(Y) \rtimes \Lambda$ has uncountably many non stably conjugate Cartan subalgebras.

Put $\Gamma = \Lambda * \mathbb{Z}$ and consider the *induced* action $\Gamma \curvearrowright (X, \mu)$ with $X = \mathrm{Ind}_\Lambda^\Gamma Y$. Observe that $\Gamma \curvearrowright (X, \mu)$ is an infinite measure preserving free ergodic action. Write $\mathcal{M} = \mathrm{L}^\infty(X) \rtimes \Gamma$ for the corresponding group measure space type II_∞ factor. Since $\Gamma = \Lambda * \mathbb{Z}$, we canonically have $\mathcal{M} = \mathcal{M}_1 *_\mathcal{B} \mathcal{M}_2$ with $\mathcal{B} = \mathrm{L}^\infty(X)$, $\mathcal{M}_1 = \mathcal{B} \rtimes \Lambda$ and $\mathcal{M}_2 = \mathcal{B} \rtimes \mathbb{Z}$. On the other hand, we also have

$$\mathcal{M} = (\mathrm{L}^\infty(Y) \rtimes \Lambda) \mathbin{\overline{\otimes}} \mathbf{B}(\ell^2(\Gamma/\Lambda)) = N \mathbin{\overline{\otimes}} \mathbf{B}(\ell^2(\Gamma/\Lambda)).$$

Therefore we obtain the following result.

Theorem III.8.1. *The amalgamated free product type* II_∞ *factor* $\mathcal{M} = \mathcal{M}_1 *_\mathcal{B} \mathcal{M}_2$ *has uncountably many nonconjugate Cartan subalgebras.*

This result shows that the condition in Theorem III.D imposing recurrence of the action $\Gamma_i \curvearrowright (X, \mu)$ for all $i \in \{1, 2\}$, is indeed necessary.

Chapter IV

Maximal amenable subalgebras of von Neumann algebras associated with hyperbolic groups

This chapter is based on a joint work with Alessandro Carderi [BC(13)]. We prove that for any infinite, maximal amenable subgroup H in a hyperbolic group G, the von Neumann subalgebra LH is maximal amenable inside LG. It provides many new, explicit examples of maximal amenable subalgebras in II_1 factors. We also prove similar maximal amenability results for direct products of relatively hyperbolic groups and orbit equivalence relations arising from measure-preserving actions of such groups.

Contents

IV.1 Introduction

Hyperfinite von Neumann algebras form the simplest and most fundamental class of von Neumann algebras. This class is very well understood: Murray and von Neumann proved that there is a unique hyperfinite II_1 factor and Connes celebrated result [Co76] states that hyperfinite von Neumann algebras are exactly the amenable ones. This characterization implies in particular that all von Neumann subalgebras of a hyperfinite tracial von Neumann algebra are completely described: they are hyperfinite. Up to now, such an understanding of subalgebras is out of reach for a non-hyperfinite von Neumann algebra.

Thus given a II_1 factor, it is natural to study the structure of its hyperfinite subalgebras. In the sixties, Kadison adressed a general question: is any self-adjoint element in a II_1 factor M contained in a hyperfinite subfactor of M? A first answer to this question was provided by Popa, who showed [Po83] that the von Neumann subalgebra of $L\mathrm{F}_n$ $(n \geq 2)$ generated by one of the generators of F_n is maximal amenable, and yet it is abelian.

Recently there has been some further work in this direction. In 2006, Shen [Sh06] extended the work of Popa to (countable) direct products of free group factors, providing the first example of an abelian maximal amenable subalgebra in McDuff factor. Subsequently, the authors in [CFRW10] investigated the radial subalgebra of $L\mathrm{F}_n$ and they managed to prove that it is also maximal amenable. Then Jolissaint [Jo(10)] extended Popa's result, providing examples of maximal amenable subalgebras in factors associated to amalgamated free-product groups, over finite subgroups. Infinitely many explicit examples of maximal amenable subalgebras were also discovered by Houdayer [Ho(12)a]. He showed that the measure class on $\mathbb{T}^2$ associated to a maximal amenable, abelian subalgebra in a II_1 factor reaches a wide range. An example from subfactor theory was also provided in [Br(12)].

In this article, we intend to provide examples of maximal amenable subalgebras of factors associated with hyperbolic groups. At the group level, amenable subgroups of hyperbolic groups are completely understood: they are virtually cyclic, and they act in a nice way on the Gromov boundary of the group. At the level of von Neumann algebras, we can show the following, generalizing the main result of [Po83].

Theorem IV.A. *Consider a hyperbolic group G and an infinite, maximal amenable subgroup $H < G$. Then the group von Neumann algebra LH is maximal amenable inside LG.*

This answers a question of Cyril Houdayer [Ho13, Problème 3.13].

Since any maximal amenable subgroup H of a hyperbolic group is virtually cyclic, the associated von Neumann algebra LH is far from being a factor. By Remark IV.3.5, we obtain many counter-examples to Kadison's question, even in property (T) factors. For instance factors of the form $L\Gamma$, with Γ a cocompact lattice in $\mathrm{Sp}(n,1)$, are counter examples with property (T).

Using similar techniques, we can prove the following result for relatively hyperbolic groups.

Theorem IV.B. *Let G be a group which is hyperbolic relative to a family $\mathcal{G}$ of subgroups of G and consider an infinite amenable subgroup $H \in \mathcal{G}$. Then the group von Neumann algebra LH is maximal amenable inside LG.*

In particular, a subgroup $H \in \mathcal{G}$ of G as in Theorem IV.B is itself maximal amenable inside G (of course there are more elementary ways to see this fact). Using results of Osin [Os06a, Os06b], we obtain the following corollary, which generalizes Theorem IV.A and the main result of [Jo(10)].

Corollary IV.C. *Let G be a group which is hyperbolic relative to a family $\mathcal{G}$ of amenable subgroups and H be an infinite maximal amenable subgroup of G. Then the group von Neumann algebra LH is maximal amenable inside LG.*

Limit groups are examples of groups G covered by this corollary.

It is also possible to prove similar results in the context of hyperbolically embedded subgroups, in the sense of [DGO(11)]: generalizing our techniques one can show that if $H < G$ is an infinite amenable subgroup which is hyperbolically embedded then LH is maximal amenable inside LG.

Finally, we extend our results to products of groups as above. We also allow the groups to act on an amenable von Neumann algebra, and we get a similar result about the crossed product von Neumann algebra. Such a product situation were already investigated in [Sh06] and [CFRW10]. We thank Stuart White for suggesting us to study this case.

Theorem IV.D. *Let $n \geq 1$, and consider for all $i = 1, \dots, n$ an inclusion of groups $H_i < G_i$ as in Theorem IV.B. Put $G := G_1 \times \cdots \times G_n$ and $H := H_1 \times \cdots \times H_n$.*

Then for any trace-preserving action of G on a finite amenable von Neumann algebra (Q, τ), the crossed-product $Q \rtimes H$ is maximal amenable inside $Q \rtimes G$.

In particular, when G and H are as above, for any free measure preserving action on a probability space $G \curvearrowright (X, \mu)$, the equivalence relation on (X, μ) given by the H-orbits is maximal hyperfinite inside the equivalence relation given by the G-orbits.

Strategy of proof

Given an inclusion $H < G$ as in Theorem IV.A or IV.B, we will analyse LH-central sequences to deduce that LH is maximal Gamma inside LG. This approach is in the spirit of Popa's asymptotic orthogonality property [Po83].

To that aim, we need to understand the conjugacy action of H on G. Once this is achieved, one can easily conclude as explained in Section IV.2.1.

In Theorem IV.D, note that $Q \rtimes H \subset Q \rtimes G$ is not maximal Gamma in general. We will in fact use Houdayer's relative version of the asymptotic orthogonality property to conclude ([Ho(12)b]). The argument relies on the same analysis of LH-central sequences.

IV.2 Preliminaries

IV.2.1 Central sequences and group von Neumann algebras

In this section, we consider an inclusion of two countable discrete groups $H < G$. We denote by $LH \subset LG$ the associated von Neumann algebras and by u_g the canonical unitaries in LG that correspond to elements $g \in G$.

For a set $F \subset G$, we will by denote $P_F : \ell^2(G) \to \ell^2(F)$ the orthogonal projection onto $\ell^2(F)$.

As explained in the introduction, the proofs of our main results rely on an analysis of LH-central sequences. We describe here how the H-conjugacy action on G allows localizing the Fourier coefficients of LH-central sequences in terms of projections P_F, $F \subset G$.

Definition IV.2.1. Let $H < G$ be an inclusion of two countable groups. A set $F \subset G \setminus H$ is said to be *H-roaming* if there exists an infinite sequence $(h_k)_{k \geq 0}$ of elements in H such that

$$h_k F h_k^{-1} \cap h_{k'} F h_{k'}^{-1} = \emptyset \text{ for all } k \neq k'.$$

Such a sequence $(h_k)_k$ is called a *disjoining sequence*.

The following standard lemma is the key of our proofs.

Lemma IV.2.2. *Let $H < G$ be an inclusion of two countable groups and denote by $LH \subset LG$ the associated von Neumann algebras. Assume that $(x_n)_n$ is a bounded LH-central sequence in LG.*

Then for any H-roaming set F we have that $\lim_n \|P_F(x_n)\|_2 = 0$.

Proof. Assume that F is an H-roaming set and consider a disjoining sequence $(h_k)_k \subset H$ for F. Since $(x_n)_n$ is LH-central, we have for all k

$$\limsup_n \|P_F(x_n)\|_2 = \limsup_n \|P_F(u_{h_k} x_n u_{h_k}^*)\|_2 = \limsup_n \|P_{h_k^{-1} F h_k}(x_n)\|_2. \tag{IV.1}$$

But $P_{h_k^{-1} F h_k}(x_n) \perp P_{h_{k'}^{-1} F h_{k'}}(x_n)$ for all $k \neq k'$ and all n. Thus we get that for any $N \geq 0$ and $n \geq 0$,

$$\|x_n\|_\infty^2 \geq \|x_n\|_2^2 \geq \sum_{k \leq N} \|P_{h_k^{-1} F h_k}(x_n)\|_2^2.$$

Applying IV.1, we deduce that $\sup_n \|x_n\|_\infty^2 \geq N \limsup_n \|P_F(x_n)\|_2^2$. Since N can be arbitrarily large, we get the result. □

Proposition IV.2.3. *Let $H < G$ be an inclusion of two infinite countable groups. Assume that for any $s, t \in G \setminus H$, there exists an H-roaming set $F \subset G \setminus H$ such that $sF^c t \cap F^c$ is finite.*

If LH has property Gamma, then it is maximal Gamma inside LG.

Proof. Assume that there exists an intermediate von Neumann algebra P with property Gamma: $LH \subset P \subset LG$. Since H is infinite, P is diffuse and so it admits a central sequence $(v_n)_n$ of unitary elements which tends weakly to 0.

Claim. For every $a \in LG \ominus LH$, we have $\lim_n \langle a v_n a^*, v_n \rangle = 0$.

By a standard linearity/density argument, to prove this claim it is sufficient to check that for all $s, t \notin H$, we have $\lim_n \langle u_s v_n u_t, v_n \rangle = 0$.

So fix $s, t \in G \setminus H$. By assumption there exists an H-roaming set F such that $K := sF^c t \cap F^c$ is finite. Since $(v_n)_n$ is LH-central and bounded, Lemma IV.2.2 implies that $\lim_n \|P_F(v_n)\|_2 = 0$. Noting that $u_s P_{F^c}(v_n) u_t$ is in the range of $P_{sF^c t}$ for all n, we obtain

$$\begin{aligned}
\limsup_n |\langle u_s v_n u_t, v_n \rangle| &= \limsup_n |\langle u_s P_{F^c}(v_n) u_t, P_{F^c}(v_n) \rangle| \\
&= \limsup_n |\langle u_s P_{F^c}(v_n) u_t, P_{sF^c t} \circ P_{F^c}(v_n) \rangle| \\
&\leq \limsup_n \|P_K(v_n)\|_2.
\end{aligned}$$

This last term is equal to 0 because $(v_n)_n$ tends weakly to 0 and K is finite, which proves the claim.

The claim implies that $P = LH$. Indeed, if $a \in P \ominus LH$, then on the one hand $\lim_n \langle a v_n a^*, v_n \rangle = \|a\|_2^2$ because the unitaries v_n asymptotically commute with a. On the other hand, this limit is equal to 0 by the claim. So $a = 0$ and we are done. □

Remark IV.2.4. By [Co76], diffuse amenable von Neumann algebras have property Gamma. Hence an amenable maximal Gamma subalgebra of a finite von Neumann algebra M is maximal amenable. In the case where $M = LG$ for some hyperbolic group G then the two notions are equivalent, because M is solid ([Oz04]).

If $H < G$ is an inclusion satisfying the assumption of Proposition IV.2.3, then H is *almost malnormal* in G in the sense that $sHs^{-1} \cap H$ is finite for all $s \notin H$. As pointed out in Example A.1.5, this is equivalent to saying that the inclusion $LH \subset LG$ is mixing (Definition A.1.2).

IV.2.2 Relatively hyperbolic groups and their boundary

The contents of this section is taken from Bowditch [Bow12]. Let us fix first some terminology and notations about graphs.

Let K be a connected graph. Its vertex set and edge set are denoted by $V(K)$ and $E(K)$ respectively. A *path* of length n between two vertices x and y is a sequence $(x_0, x_1, \dots, x_n)$ of vertices such that $x_0 = x$ and $x_n = y$, and $(x_i, x_{i+1}) \in E(K)$ for all $i = 0, \dots, n-1$. The path $(x_0, \dots, x_n)$ is a *circuit* if $x_0 = x_n$ and if $x_0, x_1, \dots, x_{n-1}$ are pairwise distinct[1]. For a path $\alpha = (x_0, x_1, \dots, x_n)$, we put $\alpha(k) = x_k$, $k = 0, \dots, n$.

We endow K with the distance d given by the length of a shortest path between two points. A path α between two vertices x and y is a *geodesic* if its length equals $d(x, y)$. We denote by $\mathcal{F}(x, y)$ the set of all geodesics between x and y.

More generally, for $r \geq 0$, a path α is an r*-quasi-geodesic* if all its vertices are distincts[2], and if for any finite subpath $\beta = (x_0, \dots, x_n)$ of α, the length of β is smaller than $d(x_0, x_n) + r$. Note that the geodesics are exactly the 0-quasi-geodesics. For $x, y \in V(K)$, denote by $\mathcal{F}_r(x, y)$ the set of r-quasi-geodesics between x and y.

We will also consider infinite paths $(x_0, x_1, \dots)$ or bi-infinite paths $(\dots, x_{-1}, x_0, x_1, \dots)$. For $r \geq 0$, such an infinite or bi-infinite path will be called r-quasi-geodesic if all its finite subpaths are r-quasi-geodesics.

Definition IV.2.5. In a graph K, a *geodesic triangle* is a set of three vertices $x, y, z \in V(K)$, together with geodesic paths $[x, y] \in \mathcal{F}(x, y)$, $[y, z] \in \mathcal{F}(y, z)$ and $[z, x] \in \mathcal{F}(z, x)$ connecting them. These paths are called the *sides* of the triangle.

Definition IV.2.6 (Gromov [Gr87])**.** A connected graph K is called *hyperbolic* if there exists a constant $\delta > 0$ such that every geodesic triangle in K is δ*-thin*: each side of the triangle is contained in the δ-neighbourhood of the union of the other two, namely $[x, y] \subset B([y, z] \cup [z, x], \delta)$, and similarly for the other two sides.

Two infinite quasi-geodesics in a hyperbolic graph K are *equivalent* if their Hausdorff distance is finite. The *Gromov boundary* ∂K of K is the set of equivalence classes of infinite quasi-geodesics. The endpoints of a path $\alpha = (x_0, x_1, \dots)$ in a class x of ∂K are defined to be x_0 and x. Similarly, a bi-infinite path $\alpha := (\dots, x_{-1}, x_0, x_1, \dots)$ has endpoints $\alpha_- := [(x_0, x_{-1}, \dots)] \in \partial K$ and $\alpha_+ = [(x_0, x_1, \dots)] \in \partial K$. It turns out that for any two points $x, y \in K \cup \partial K$, for any $r \geq 0$, the set $\mathcal{F}_r(x, y)$ of r-quasi-geodesics connecting them is non-empty.

Recall that a *hyperbolic group* is a finitely generated group which admits a hyperbolic Cayley graph (this implies that all its Cayley graphs are hyperbolic). We will define similarly relatively hyperbolic groups, but we have to replace the Cayley graph by a graph in which some subgroups are “collapsed” to points.

Definition IV.2.7 ([Far98])**.** Consider a group G, with finite generating set S and denote by $\Gamma := \mathrm{Cay}(G, S)$ the associated Cayley graph. Let $\mathcal{G}$ be a collection of subgroups of G. The *coned-off graph of* Γ *with respect to* $\mathcal{G}$ is the graph $\hat{\Gamma}$ with:

- vertex set $V(\hat{\Gamma}) := V(\Gamma) \sqcup \bigsqcup_{H \in \mathcal{G}} G/H$;

[1] This last condition will be crucial in Definition IV.2.8.

[2] This condition is necessary for Lemma IV.2.12(1) to be true.

- edge set $E(\hat{\Gamma}) := E(\Gamma) \sqcup \{(gh, [gH]) \mid H \in \mathcal{G}, [gH] \in G/H, h \in H\}$.

In the sequel, we will identify $V(\Gamma)$ with G. The action of G on itself by left multiplication extends to an isometric action on $\hat{\Gamma}$. The stabilizer of the vertex $[gH]$ is equal to gHg^{-1}.

Note that in the coned-off graph $\hat{\Gamma}$, the distance between two elements g and gh is at most 2 whenever $h \in H$ for some $H \in \mathcal{G}$. Note also that this coned-off graph will not be locally finite in general. But it will sometimes satisfy the following *fineness* condition.

Definition IV.2.8 ([Bow12])**.** A graph Γ is called *fine* if each edge of Γ is contained in only finitely many loops of length n, for any given integer n.

Definition IV.2.9 ([Bow12])**.** A group G is said to be *hyperbolic relative to the family $\mathcal{G}$* if there exists a finite generating set S of G such that the coned-off graph $\hat{\Gamma}$ is fine and δ-hyperbolic (for some $\delta \geq 0$).

From this definition, usual hyperbolic groups appear as hyperbolic relative to the empty family. For any relatively hyperbolic group G with Cayley graph Γ, let us define a topology on $\Delta\Gamma := \hat{\Gamma} \cup \partial\hat{\Gamma}$.

Definition IV.2.10. Given $x \in \Delta\Gamma$ and a finite set $A \subset V(\hat{\Gamma})$ such that $x \notin A$, we define

$$M(x, A) := \{y \in \Delta\Gamma : A \cap \alpha = \emptyset, \forall \alpha \in \mathcal{F}(x, y)\}.$$

Theorem IV.2.11 ([Bow12], section 8)**.** *The family $\{M(x, A)\}_{x,A}$ is a basis for a Hausdorff compact topology on $\Delta\Gamma$ such that $G \subset \Delta\Gamma$ is a dense subset, and every graph automorphism of $\hat{\Gamma}$ extends to a homeomorphism of $\Delta\Gamma$.*

Actually, we will not use the fact that $\Delta\Gamma$ is compact. The proof of Theorem IV.2.11 relies on the following lemma, which will be our main tool in order to manipulate neighbourhoods in $\Delta\Gamma$.

Lemma IV.2.12 ([Bow12], Section 8)**.** *Let $r \geq 0$. The following facts are true.*

1. *For every $x, y \in \Delta\Gamma$, the graph $\bigcup_{\alpha \in \mathcal{F}_r(x,y)} \alpha$ is locally finite.*
2. *For every edge $e \in E(\hat{\Gamma})$, there exists a finite set $E_r(e) \subset E(\hat{\Gamma})$ such that for all $x, y \in \Delta\Gamma$, and all $\alpha, \beta \in \mathcal{F}_r(x, y)$ with $e \in \alpha$, we have that $E_r(e) \cap \beta$ contains at least one edge.*
3. *For every $a \in V(\hat{\Gamma}), x \in \Delta\Gamma$, with $x \neq a$, there exists a finite set $V_{r,x}(a) \subset V(\hat{\Gamma}) \setminus \{x\}$ such that for all $y \in \Delta\Gamma$, and all $\alpha, \beta \in \mathcal{F}_r(x, y)$ with $a \in \alpha$, we have that $\beta \cap V_{r,x}(a) \neq \emptyset$.*

More generally, given a subset $A \subset V(\hat{\Gamma})$, we will put $V_{r,x}(A) := \bigcup_{a \in A} V_{r,x}(a)$. We will also assume that $A \subset V_{r,x}(A)$.

Proof. The first two facts are 8.2 and 8.3 in [Bow12]. To derive the third fact from the others, fix $a \in V(\hat{\Gamma})$ and $x \in \Delta\Gamma$. Denote by E_0 the set of edges e in the graph $\bigcup_{\alpha \in \mathcal{F}_r(a,x)} \alpha$ such that a is an endpoint of e. By (1), the set E_0 is finite. Now put $E := \bigcup_{e \in E_0} E_r(e)$, and define $V_{r,x}(a)$ to be the set of endpoints of E, in which we remove x if necessary. This is a finite set.

Now if $\alpha \in \mathcal{F}_r(x, y)$ goes through a, then it will contain an edge in E_0. Thus any $\beta \in \mathcal{F}_r(x, y)$ contains an edge in E, and we are done by the definition of $V_{r,x}(a)$. ☐

Lemma IV.2.12 will always be used via the following easy lemma.

Lemma IV.2.13. *Let $r > 0$, $x \in \Delta\Gamma$ and $A \subset V(\hat{\Gamma}) \setminus \{x\}$ finite. The following are true.*

1. *If $y \notin M(x, A)$, then any r-quasi-geodesic $\alpha \in \mathcal{F}_r(x, y)$ intersects $V_{r,x}(A)$.*
2. *If $y \in M(x, V_{r,x}(A))$, then no r-quasi-geodesic from y to x intersect A.*

Now we describe a way of constructing quasi-geodesic paths. The following lemma is well known.

Lemma IV.2.14. *There exist a constant $r_0 \geq 0$, only depending on the hyperbolicity constant of the graph $\hat{\Gamma}$, with the following property: for any geodesic paths α, β sharing exactly one endpoint a, if a is the closest point of α to each point of β, then $\alpha \cup \beta$ is an r_0-quasi-geodesic.*

Definition IV.2.15. Consider $x, y, z \in \Delta\Gamma$ and let $\alpha \in \mathcal{F}(x, y)$ be a geodesic. A point $z_0 \in \alpha$ which minimizes the distance from z to α, is called a *projection of z on α*. Such a z_0 splits the path α into two geodesic paths $\alpha_x \in \mathcal{F}(x, z_0)$ and $\alpha_y \in \mathcal{F}(z_0, y)$. Given any geodesic $\beta \in \mathcal{F}(z, z_0)$, we can join β and α_x or α_y to get two paths that are r_0-quasi geodesic by Lemma IV.2.14.

We end this section with a lemma that we will need later. Its proof illustrates well how to use the tools defined above.

Lemma IV.2.16 ([Bow12])**.** *For every $x \in \Delta\Gamma$ and for every finite subset $A \subset V(\hat{\Gamma}) \setminus \{x\}$, there exists a finite subset $C \subset V(\hat{\Gamma}) \setminus \{x\}$ such that for every $y \in M(x, C)$,*

$$M(y, C) \subset M(x, A).$$

Proof. Let $r_0 \geq 0$ be given by Lemma IV.2.14, and set $C := V_{r_0,x}(V_{r_0,x}(A))$ (see Lemma IV.2.12(3)). We will show that the conclusion of the lemma holds for this C.

If $y = x$, we see that $M(x, C) \subset M(x, A)$ because $A \subset V_{r_0,x}(A) \subset C$. Now let $y \in M(x, C)$, with $y \neq x$, and take $z \notin M(x, A)$. We will show that $z \notin M(y, C)$.

Let α be a geodesic between y and z. Consider a projection x_0 of x on α as in Definition IV.2.15 and let $\beta \in \mathcal{F}(x, x_0)$. We denote with α_y (resp. α_z) the subgeodesic of α between x_0 and y (resp. x_0 and z). Then, by Lemma IV.2.14 the paths $\beta \cup \alpha_y \in \mathcal{F}_{r_0}(x, y)$ and $\beta \cup \alpha_z \in \mathcal{F}_{r_0}(x, z)$ are r_0-quasi-geodesics.

Since $z \notin M(x, A)$, Lemma IV.2.13(1) implies that $\beta \cup \alpha_z$ intersects $V_{r_0,x}(A)$. If the intersection point lied on $\beta \cup \alpha_y$, then Lemma IV.2.13(2) would contradict our assumption that $y \in M(x, C)$. Hence the intersection point lies on $\alpha_z \subset \alpha$. We have found a geodesic between z and y which intersects a point of $V_{r_0,x}(A) \subset C$, which means precisely that $z \notin M(y, C)$. □

IV.3 Hyperbolic case: proof of Theorem IV.A

Suppose that G is a hyperbolic group and that H is an infinite maximal amenable subgroup of G. We want to apply Proposition IV.2.3 in order to prove Theorem IV.A.

As mentioned in Section IV.2.2, G is hyperbolic relative to the empty family and $\hat{\Gamma} = \Gamma$, for any Cayley graph Γ of G. Thus $\Delta\Gamma := \Gamma \cup \partial\Gamma$ is the usual Gromov compactification of Γ, with boundary $\partial\Gamma$, endowed with the topology given by the sets $\{M(x,A)\}_{x,A}$. As before, we identify G with $V(\Gamma)$.

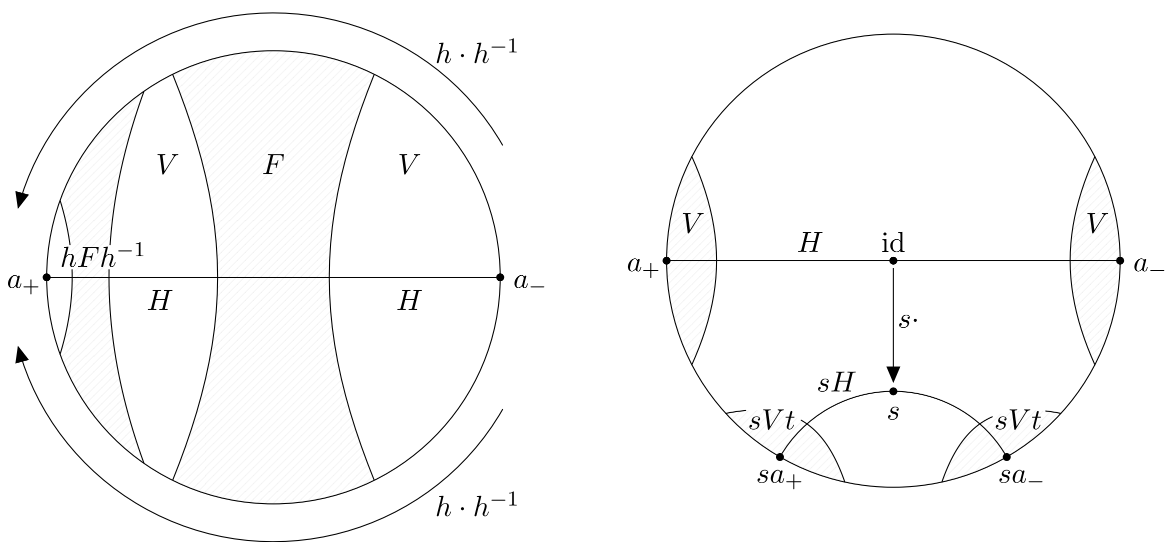
Recall that the action of G by left multiplication on itself extends to a continuous action on $\Delta\Gamma$. This action is amenable.

The amenable subgroup H has a particular form by [GdH90, Théorème 8.29, Théorème 8.37]. First, H admits an element $a \in H$ which generates a finite index subgroup of H. Second, the element a acts on $\Delta\Gamma$ with exactly two fixed points $a_\pm \in \partial\Gamma$, and $H \subset \mathrm{Stab}_G(\{a_-, a_+\})$. Since the left action of G on $\Delta\Gamma$ is amenable, the group $\mathrm{Stab}_G(\{a_-, a_+\})$ is amenable. By maximal amenability of H this yields the equality $H = \mathrm{Stab}_G(\{a_-, a_+\})$. Also, $\mathrm{Stab}(a_-)$ and $\mathrm{Stab}(a_+)$ are contained in H.

Moreover, the fixed points $a_\pm$ of a are such that $\lim_{n\to+\infty} a^n x = a_+$ and $\lim_{n\to-\infty} a^n x = a_-$, for any $x \in \Delta\Gamma$ (so in particular a_+ is the unique cluster point of the sequence $\{a^n\}_{n\geq 0}$).

The action of G on itself by right multiplication also extends to a continuous action on $\Delta\Gamma$, in such a way that any element $g \in G$ acts trivially on $\partial\Gamma$ (see for instance [BO08, Proposition 5.3.18]).

In order to find an H-roaming set as in Proposition IV.2.3, we need to understand geometrically the conjugacy action of H on G. We start by collecting properties of left and right actions of H on $\Delta\Gamma$ separately, in the following two lemmas. Combining these lemmas, we will see that the conjugacy action of H has a uniform *"north-south dynamics" out of* H, as shown in Figure IV.1a.



(a) Conjugacy action of H on G.

(b) The subsets V and sVt are disjoint.

Figure IV.1: The action of G and a good neighborhood V of $\{a_+, a_-\}$.

The following fact is certainly known, but we include a proof for the sake of completeness.

Lemma IV.3.1. *For any finite sets $A, B \subset V(\Gamma)$, there exists $n \in \mathbb{Z}$ such that*

$$G \cap (a^n \cdot M(a_-, B)^c) \subset M(a_+, A).$$

Proof. First note that we can (and we will) assume that $a_- \notin M(a_+, A)$. By Lemma IV.2.16 there exists a finite set $C \subset V(\Gamma)$ such that for all $y \in M(a_+, C)$ we have $M(y, C) \subset M(a_+, A)$.

In particular, for all $y \in M(a_+, C)$ and $z \notin M(a_+, A)$ there exists a geodesic between y and z which intersects C.

Choose $n \in \mathbb{Z}$ such that $a^n B \subset M(a_+, C)$ and such that the distance between points of C and $a^n B$ is larger than the diameter D of $V_{0,a_-}(C)$. We claim that this n satisfies the conclusion of the lemma.

Assume by contradiction that there exists $z \in G = V(\Gamma)$ such that $z \notin a^n M(a_-, B)$ and $z \notin M(a_+, A)$. Since $z \notin a^n M(a_-, B) = M(a_-, a^n B)$, there exists a geodesic $\alpha \in \mathcal{F}(a_-, z)$ which contains a point $y \in a^n B \subset M(a_+, C)$. Let us denote α_{a_-} the sub-geodesic of α between a_- and y and with α_z the sub-geodesic between y and z.

Since $a_- \notin M(a_+, A)$, there exists a geodesic between a_- and y which intersects C. By Lemma IV.2.12, the geodesic α_{a_-} meets $V_{0,a_-}(C)$ at a vertex x_1. Moreover $z \notin M(a_+, A)$, so replacing α_z by another geodesic between y and z if necessary, we can assume that α_z meets $C \subset V_{0,a_-}$ at a vertex x_2 (while $\alpha = \alpha_{a_-} \cup \alpha_z$ is still a geodesic). But then

$$d(x_1, x_2) \leq \operatorname{diam}(V_{0,a_-}(C)) = D.$$

On the other hand, the length of α between these two points is equal to $d(x_1, y) + d(y, x_2)$, while $d(x_1, y) > D$ because $x_1 \in C$ and $y \in a^n B$. This is absurd. $\square$

Lemma IV.3.2. *For any $A \subset V(\Gamma)$ finite, there exists a finite $B \subset V(\Gamma)$ such that for any $k \in \mathbb{Z}$,*

$$(M(a_+, B) \cap (G \setminus H))a^k \subset M(a_+, A).$$

Proof. We start with a claim.

Claim. There exists a finite set $B' \subset V(\Gamma)$ such that if $y \in M(a_+, B') \cap G$ is such that $ya^k \notin M(a_+, A)$ for some $k \in \mathbb{Z}$, then there exists $m \in \mathbb{Z}$ such that $ya^m \in B'$.

Proof of the Claim. By [GdH90, Proposition 8.21], there exists a finite constant $r > 0$ such that for any $p \in \mathbb{Z}$, all geodesics between the neutral element e and a^p are contained in the r-neighbourhood of the sequence $\{a^k, k \in \mathbb{Z}\}$.

By Lemma IV.2.16 there exists a finite set $C \subset V(\Gamma)$ such that for all $y \in M(a_+, C)$ we have $M(y, C) \subset M(a_+, A)$. We show the claim for $B' := B(C, r)$, the r-neighbourhood of C.

Take $y \in M(a_+, B') \cap G \subset M(a_+, C)$ such that $ya^k \notin M(a_+, A)$ for some $k \in \mathbb{Z}$. Then $ya^k \notin M(y, C)$, so there exists a geodesic α between y and ya^k which meets C at a point c. Then $y^{-1}c$ belongs to a geodesic between e and a^k, so it is at distance less than r to some a^m. In other words, $ya^m \in B(C, r) = B'$, which proves the claim. $\square$

Observe that the set of cluster points of the sequences $(ya^k)_k$, with $y \in B' \setminus H$ is finite and contained in $\partial\Gamma \setminus \{a_+, a_-\}$. So there exists B such that

$$M(a_+, B) \subset M(a_+, B') \quad \text{and} \quad M(a_+, B) \cap \{ba^k \,|\, b \in B' \setminus H, k \in \mathbb{Z}\} = \emptyset.$$

The subset B satisfies the conclusion of the lemma. Indeed, if $y \in M(a_+, B) \cap (G \setminus H)$ is such that $ya^k \notin M(a_+, A)$ for some $k \in \mathbb{Z}$, then by the claim there exists h such that $y \in B'a^{-h}$. But in this case we would have $y \in M(a_+, B) \cap \{ba^p \,|\, b \in B' \setminus H, p \in \mathbb{Z}\}$, which was assumed to be empty. Therefore $ya^k \in M(a_+, A)$ for any k. $\square$

Now we can deduce a relevant property of the conjugacy action of H, as shown in Figure IV.1a.

Proposition IV.3.3. *For any neighbourhood V of $\{a_-, a_+\}$ inside $\Delta\Gamma$, the set $F := V^c \cap (G \setminus H)$ is H-roaming.*

Proof. Let V and F be as in the proposition. We will construct a disjoining sequence $(h_k)_k$ inductively. First put $h_0 := e$.

Now assume that $h_0, \dots, h_{n-1}$ have been constructed, for some $n \geq 1$. We will construct h_n. Denote by $V_n := \bigcap_{i=0}^{n} h_i V h_i^{-1}$. It is a neighbourhood of $\{a_-, a_+\}$, by continuity of left and right actions of H. Now put $F_n := V_n^c \cap (G \setminus H)$.

By Lemma IV.3.2, there exists a neighbourhood V' of a_+ such that $(V' \cap (G \setminus H))a^k \subset V_n$ for all $k \in \mathbb{Z}$. By Lemma IV.3.1, there exists $k_n \in \mathbb{Z}$ such that $G \cap a^{k_n} V^c \subset V'$ and in particular $a^{k_n} F \subset V'$. Note also that $(a^{k_n} F) \cap H = \emptyset$.

Altogether, we get that $a^{k_n} F a^{-k_n} \subset V_n$ is disjoint from F_n. But F_n contains all the $h_i F h_i^{-1}$, $i \leq n-1$. So we can define $h_n = a^{k_n}$. $\square$

Corollary IV.3.4. *For every $s, t \in G \setminus H$, there exists an H-roaming set $F \subset G \setminus H$ such that $sF^c t \cap F^c$ is finite.*

Proof. Choose a neighbourhood V_0 of $\{a_+, a_-\}$ such that V_0 is disjoint from sV_0. Since the right action of t on $\Delta\Gamma$ is continuous, we can find a $V \subset V_0$ such that V and sVt are disjoint (see Figure IV.1b). We observe that $sVt \cap H$, $sHt \cap V$ and $sHt \cap H$ are finite because the only cluster points of H are in V and the only cluster points of sHt are in sVt.

Therefore, setting $F := V^c \cap (G \setminus H)$, we get an H-roaming set (by Proposition IV.3.3) such that $sF^c t \cap F^c$ is finite. $\square$

Now Theorem IV.A follows from Proposition IV.2.3.

Remark IV.3.5. Note that in the proof of Proposition IV.3.3 the disjoining sequence that we construct is contained in the subgroup $H_0 := \langle a \rangle \subset H$. Then the proof of Theorem IV.A actually shows that if $P \subset LG$ is an algebra with property Gamma such that $LH_0 \subset P$, then $P \subset LH$. Hence u_a is contained in a unique maximal amenable von Neumann subalgebra of M.

IV.4 Relatively hyperbolic case

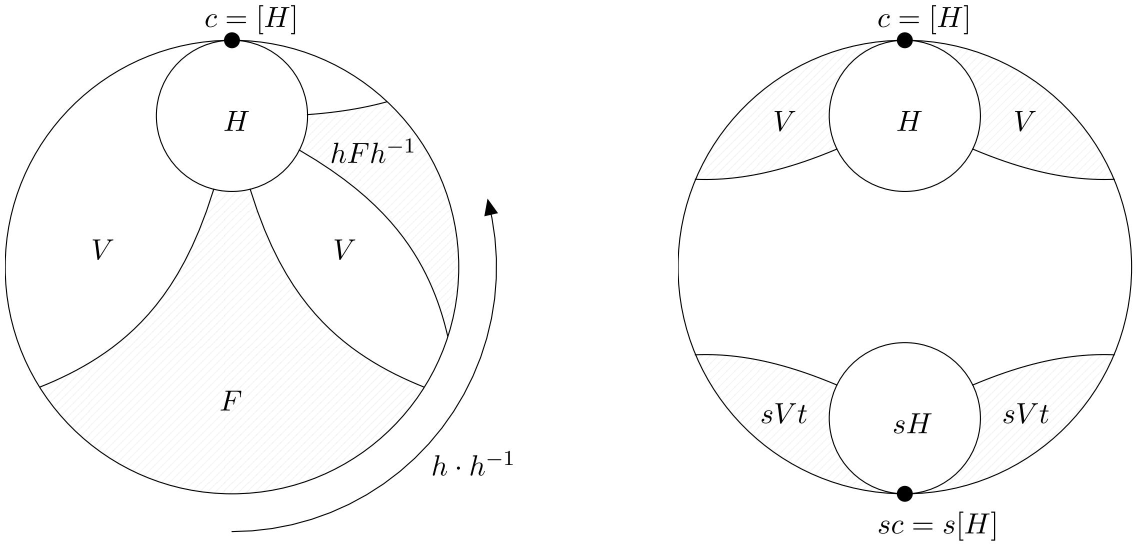
IV.4.1 Proof of Theorem IV.B

Let G be a hyperbolic group relative to a family $\mathcal{G}$ of subgroups of G, and let $H \in \mathcal{G}$ be an infinite amenable subgroup.

Consider a Cayley graph Γ of G such that the coned-off graph $\hat{\Gamma}$ of Γ with respect to $\mathcal{G}$ is fine and hyperbolic. Denote by $\Delta\Gamma$ its Gromov compactification, endowed with the topology generated by the sets $\{M(x, A)\}_{x,A}$. We still identify G with the subset $V(\Gamma) \subset V(\hat{\Gamma})$.

Now denote by $c = [H] \in V(\hat{\Gamma})$ the vertex associated with $[H] \in G/H$. This point is not in the boundary $\partial\Gamma$, but it is represented out of Γ, as in Figure IV.1.

We will show that for any neighbourhood V of c, the set $F := V^c \cap (G \setminus H)$ (Figure IV.2a) is H-roaming in the sense of Definition IV.2.1. Then we will show that if V is small enough (Figure IV.2b), F satisfies the condition of Proposition IV.2.3, hence proving Theorem IV.B.



(a) Conjugacy action of H on G. (b) The subsets V and sVt are disjoint.

Figure IV.2: The action of G and a good neighborhood V of $c = [H]$.

In this section, we will write V_r instead of $V_{r,c}$, $r \geq 0$ (see Lemma IV.2.12).

Remark that since c shares an edge with all the points in H (and only with them), any geodesic between c and a point $x \in \Delta\Gamma$ contains exactly one element in H. In particular one has the following simple lemma.

Lemma IV.4.1. *The family $\{M(c, A)\}_{A \subset H}$ is a basis of neighbourhoods of c.*

Proof. Let $B \subset V(\hat{\Gamma})$ be a finite subset, for every $b \in B$ choose a geodesic α_b from c to b. Set $A := \{\alpha_b(1)\}_{b \in B}$ and observe that $M(c, A) \subset M(c, B)$. □

Remark IV.4.2. In the same way, if $A \subset H$ is finite and $r \geq 0$, the set $V_r(A)$ from Lemma IV.2.12 can be assumed to be contained in H. Indeed one can replace $V_r(A)$ by the finite set of points in H which lie on an r quasi-geodesic from $V_r(A)$ to c.

To give a hint about the topology near the point c, let us mention that any sequence $(h_n)_n$ in H which goes to infinity converges to c.

As in the hyperbolic case, we will study geometrically the conjugacy action of H on G. We will treat left and right actions separately. First, the left multiplication of G on itself extends to an isometric action on $\hat{\Gamma}$, and hence extends to a continuous action on $\Delta\Gamma$. Let us extend also the right action.

Definition IV.4.3. The *right action* of G on $\Delta\Gamma$ is the action whose restriction to G is equal to the right multiplication by G on itself, and which is trivial on $\Delta\Gamma \setminus G$. This action is *a priori* not continuous, and it clearly commutes with the left action.

The following lemma is due to Ozawa, who shows actually that the right action on $\Delta\Gamma$ is continuous. But we will only use continuity at c.

Lemma IV.4.4 ([Oz06])**.** *The right action of G on $\Delta\Gamma$ is continuous at c.*

Proof. Let $g \in G$, and let (x_n) be a sequence converging to c. We want to prove that $(x_n g)$ converges to c. Since the right action is trivial on $\Delta\Gamma \setminus G$, we can assume that $x_n \in G$ for all n. Fix a finite set $A \subset H$. We will show that $x_n g \in M(c, A)$ for n large enough.

By Lemma IV.2.16, there exists a finite set $C \subset V(\hat{\Gamma}) \setminus \{c\}$ such that for all $y \in M(c, C)$ we have $M(y, C) \subset M(c, A)$. So if $y \in M(c, C)$ and $z \notin M(c, A)$, then there exists a geodesic between y and z which intersects C.

Assume by contradiction that there exist infinitely many indices n for which $x_n g \notin M(c, A)$. By assumption $x_n \in M(c, C)$ for n large enough, which implies that there exists a geodesic $\alpha_n \in \mathcal{F}(x_n, x_n g)$ which intersects C for infinitely many n's. Then $x_n^{-1}\alpha_n$ belongs to $\mathcal{F}(e, g)$ and the set $X := \bigcup_{\alpha \in \mathcal{F}(e,g)} V(\alpha)$ is finite by Lemma IV.2.12(1). Altogether we get that $x_n^{-1} C \cap X \neq \emptyset$ for infinitely many n's. Taking a subsequence if necessary, we find an element $c' \in C$ and $x \in X$ such that $x_n^{-1} c' = x$ for all n.

But then for all $p, n \in \mathbb{N}$, we see that $x_p^{-1} x_n \in \mathrm{Stab}_G(x)$. Since there are infinitely many distinct elements x_n, we get that x has to be a conic point, and for all fixed p, the sequence $(x_p^{-1} x_n)_n$ converges to x. But by continuity of the left action, the sequence also converges to $x_p^{-1} c$.

Therefore $c = x_p x = c'$. This contradicts our assumption that $c \notin C$. □

We now collect properties of left and right actions of H on $\Delta\Gamma$. Note that the left action of H stabilizes c (and $H = \mathrm{Stab}(c)$).

Lemma IV.4.5. *For any finite subsets $A, B \subset H$, there exists $h \in H$ such that*

$$hM(c, A)^c \subset M(c, B).$$

Proof. By Remark IV.4.2, we may assume that $V_0(A) \subset H$. Let $h \in H$ be such that $hV_0(A) \cap B = \emptyset$. Let $x \in M(c, A)^c$ and let α be any geodesic between c and hx, $\alpha \in \mathcal{F}(c, hx)$. By Lemma IV.2.13(1), $h^{-1}\alpha \in \mathcal{F}(c, x)$ contains a point $a \in V_0(A)$. Thus ha is the unique point of H which is on α. In particular α contains no point of B. □

Lemma IV.4.6. *For any $A \subset V(\hat{\Gamma})$ finite, there exists a finite $B \subset V(\hat{\Gamma})$ such that for any $h \in H$,*

$$(M(c, B) \cap (G \setminus H))h \subset M(c, A).$$

Proof. By Lemma IV.4.1, we can assume that $A \subset H$. Consider an element $x \in G \setminus H$ such that $x \notin M(c, A)$ and take $h \in H$. We will show that $xh \notin M(c, V_2(A))$.

Let α be a geodesic from c to x that meets A and put $a := \alpha(1) \in \alpha \cap A$. Note that since $xh \notin H$, we have $d(xh, c) \geq 2$, and at the same time $d(xh, x) \leq 2$, because xh and x lie in the same coset xH. Hence one can choose a projection z_0 of xh on α to be different from c. Thus the path from xh to c through z_0 constructed as in Definition IV.2.15 is a 2-quasi-geodesic and it contains $a = \alpha(1) \in A$. By Remark IV.2.13(2), this implies that $xh \notin M(c, V_2(A))$. Thus $B := V_2(A)$ satisfies the conclusion of the lemma. □

As in the hyperbolic case, we deduce the following property of the conjugacy action of H on G, see Figure IV.2a.

Proposition IV.4.7. *For any neighbourhood V of c inside $\Delta\Gamma$, the set $F := V^c \cap (G \setminus H)$ is H-roaming.*

Proof. We will construct a disjoining sequence $(h_k)_k \subset H$ inductively. First put $h_0 := e$. Now assume that $h_0, \dots, h_{n-1}$ have been constructed, for some $n \geq 1$. We will construct h_n.

Denote by $V_n := \bigcap_{i=0}^n h_i V h_i^{-1}$. It is a neighbourhood of $\{c\}$, by continuity at c of left and right actions of H. Hence there exists $A \subset H$ finite such that $M(c, A) \subset V_n$. Now put $F_n := M(c, A)^c \cap (G \setminus H)$.

By Lemma IV.4.6, there exists $B \subset H$ finite such that $(M(c, B) \cap (G \setminus H))H \subset M(c, A)$. Then Lemma IV.4.5 provides an $h \in H$ such that $hF_n \subset M(c, B)$. Altogether, we get that $hF_n h^{-1} \subset M(c, A)$, which is disjoint from F_n.

Note that for all $i = 0, \dots, n-1$, we have $h_i F h_i^{-1} \subset V_n^c \cap (G \setminus H) \subset F_n$. Therefore we can define h_n to be equal to h. □

Corollary IV.4.8. *For every $s, t \in G \setminus H$, there exists an H-roaming set $F \subset G \setminus H$ such that $sF^c t \cap F^c$ is finite.*

Proof. We proceed as in Corollary IV.3.4. By continuity of left and right action at c, there exists a neighbourhood V of c such that V and sVt are disjoint (see Figure IV.2b). We observe that $sVt \cap H$, $sHt \cap V$ and $sHt \cap H$ are finite because the cluster point of H lies in V and the cluster point of sHt lies in sVt.

Therefore, setting $F := V^c \cap (G \setminus H)$, we get an H-roaming set (by Proposition IV.4.7) such that $sF^c t \cap F^c$ is finite. □

Now Theorem IV.B follows from Proposition IV.2.3.

Surprisingly, we did not use the fact that H is amenable until the end of the proof. In fact our proof shows that if G is hyperbolic relative to a family $\mathcal{G}$ of subgroup and if $H \in \mathcal{G}$ is infinite and such that LH has property Gamma then LH is maximal Gamma inside LG.

Remark IV.4.9. For later use, note that the set F in Corollary IV.4.8 is such that $sF^c t \cap F^c \subset H$ and hence $s(F \cup H)^c t \subset F \cup H$.

IV.4.2 Proof of Corollary IV.C

Assume that G is hyperbolic relative to a family $\mathcal{G}$ of amenable subgroups, and consider an infinite maximal amenable subgroup $H < G$. We will show that G is hyperbolic relative to $\mathcal{G} \cup \{H\}$. Then Theorem IV.B will directly allow to conclude that LH is maximal amenable inside LG.

The argument relies on Osin's work [Os06a, Os06b].

Definition IV.4.10. An element $g \in G$ is said to be *hyperbolic* if it has infinite order and is not contained in a conjugate of a group in $\mathcal{G}$.

Definition IV.4.11. A subgroup K of G is said to be *elementary* if it is either finite, or contained in a conjugate of a group in $\mathcal{G}$, or if it contains a finite index cyclic subgroup $\langle g \rangle$, for some hyperbolic element g.

(Gromov-)Tukia's strong Tits alternative (see [Tu94, Theorem 2T, Theorem 3A] using [Bow12, Definition 1]) states that a non-elementary subgroup K of G contains a copy of the free group on two generators.

In particular, our amenable subgroup H is elementary. If it is contained in a conjugate aH_ia^{-1} of a group in $\mathcal{G}$, then it is equal to aH_ia^{-1} by maximal amenability, and Theorem IV.B concludes.

Now assume that H contains a finite index cyclic subgroup $\langle g \rangle$, for some hyperbolic element g. Osin showed in [Os06b, Section 3] that such a hyperbolic element g is contained in a unique maximal elementary subgroup $E(g)$ (thus $H = E(g)$, by maximal amenability). Moreover he showed [Os06b, Corollary 1.7] that G is hyperbolic relative to $\mathcal{G} \cup \{E(g)\}$. This is what we wanted to show.

IV.5 Product case: proof of Theorem IV.D

Observe that if $H_i < G_i$, for $i = 1, 2$, are infinite maximal amenable subgroups, then the von Neumann subalgebra $L(H_1 \times H_2) \subset L(G_1 \times G_2)$ is neither maximal Gamma nor mixing as soon as $H_1 \neq G_1$.

Therefore to treat the product case, we will have to deal with relative notions. We could consider a relative notion of property Gamma and proceed as in Section IV.2.1. We choose instead to apply directly the work of C. Houdayer and the *relative asymptotic orthogonality property*, [Ho(12)b]. Note that in the case of virtually abelian subgroups H_1, H_2 we could also use [CFRW10, Theorem 2.8].

Definition IV.5.1. Let $A \subset N \subset (M, \tau)$ be finite von Neumann algebras. The inclusion $N \subset M$ is said to be *weakly mixing through* A if the bimodule ${}_AL^2(M) \ominus L^2(N)_N$ is (left) weakly mixing, in the sense of Definition A.2.1.

In the spirit of Example A.1.5, one can check the following.

Example IV.5.2. If $H < G$ is an inclusion of groups satisfying the assumption of Proposition IV.2.3 (*e.g.* if H and G are as in Theorem IV.B), then for any trace-preserving action $G \curvearrowright (Q, \tau)$ on a finite von Neumann algebra, the inclusion $Q \rtimes H \subset Q \rtimes G$ is weakly mixing through LH.

Definition IV.5.3 ([Ho(12)b], Definition 5.1)**.** Let $A \subset N \subset (M, \tau)$ be an inclusion of finite von Neumann algebras. We say that $N \subset M$ has the *asymptotic orthogonality property relative to* A if for every $\|\cdot\|_\infty$-bounded sequences $(x_n)_n$ and $(y_n)_n$ in $M \ominus N$ which asymptotically commute with A, we have that

$$\lim_n \langle a x_n b, y_n \rangle = 0, \text{ for all } a, b \in M \ominus N.$$

Theorem IV.5.4 ([Ho(12)b], Theorem 8.1)**.** *Let $A \subset N \subset (M, \tau)$ be an inclusion of finite von Neumann algebras. Assume the following:*

1. *A is amenable.*
2. *The inclusion $N \subset M$ is weakly mixing through A.*
3. *The inclusion $N \subset M$ has the relative asymptotic orthogonality property relative to A.*

Then any amenable von Neumann subalgebra of M containing A is automatically contained in N.

From now on, we consider the crossed-product von Neumann algebras $Q \rtimes G$ associated to a trace-preserving actions $G \curvearrowright (Q, \tau)$. As for group von Neumann algebras, denote by u_g the unitaries of $Q \rtimes G$ corresponding to elements $g \in G$ and for any set $F \subset G$ denote by $P_F : L^2(Q, \tau) \otimes \ell^2(G) \to L^2(Q, \tau) \otimes \ell^2(F)$ the orthogonal projection.

Proposition IV.5.5. *Let $H < G$ be an inclusion of two infinite groups, with H amenable. Consider an action $G \curvearrowright (Q, \tau)$ of G on a tracial von Neumann algebra, and assume that for any $s, t \in G \setminus H$, there exists an H-roaming set $F \subset G \setminus H$ such that $s(F \cup H)^c t \subset F \cup H$.*

Then the inclusion $Q \rtimes H \subset Q \rtimes G$ has the asymptotic orthogonality property relative to LH.

Proof. Consider two $\|\cdot\|_\infty$-bounded sequences $(x_n)_n$ and $(y_n)_n$ in $(Q \rtimes G) \ominus (Q \rtimes H)$ which asymptotically commute with LH. By linearity and density it is sufficient to check that for any $s, t \notin H$,

$$\lim_n \langle u_s x_n u_t, y_n \rangle = 0.$$

Fix $s, t \in G \setminus H$. There exists an H-roaming set F such that $s(F \cup H)^c t \subset F \cup H$. Proceeding as in the proof of Lemma IV.2.2, it is easy to show that $\lim_n \|P_F(x_n)\|_2 = \lim_n \|P_F(y_n)\|_2 = 0$. Note also that for all n, we have $x_n = P_{H^c}(x_n)$ and $y_n = P_{H^c}(y_n)$. Therefore

$$\begin{aligned}\lim_n \langle u_s x_n u_t, y_n \rangle &= \lim_n \langle u_s P_{F^c}(x_n) u_t, P_{F^c}(y_n) \rangle \\ &= \lim_n \langle u_s P_{(F \cup H)^c}(x_n) u_t, P_{(F \cup H)^c}(y_n) \rangle = 0,\end{aligned}$$

because $s(F \cup H)^c t \subset F \cup H$. This ends the proof of the proposition. □

Proof of Theorem IV.D. For $i = 1, \ldots, n$, let G_i be a hyperbolic group relative to a family $\mathcal{G}_i$ of subgroups and let $H_i \in \mathcal{G}_i$ be an infinite amenable group. Consider the inclusion

$$H := H_1 \times \cdots \times H_n < G := G_1 \times \cdots \times G_n.$$

Let (Q, τ) be a finite amenable von Neumann algebra and consider a trace-preserving action $G \curvearrowright (Q, \tau)$ of G. Put $N := Q \rtimes H$ and $M := Q \rtimes G$.

Assume that P is an intermediate amenable von Neumann subalgebra: $N \subset P \subset M$. We have to show that $P = N$. In order to do so, we will show that for all $i = 1, \ldots, n$, we have

$$P \subset N_i := Q \rtimes (G_1 \times \cdots \times G_{i-1} \times H_i \times G_{i+1} \times \cdots \times G_n).$$

This is enough to conclude, because $N = \cap_{i=1}^n N_i$.

For $i \in \{1, \ldots, n\}$, we set $A_i := LH_i$ and $Q_i := Q \rtimes \hat{G}_i$, where $\hat{G}_i$ is the direct product of all G_j, $j \neq i$. Then we have $N_i \simeq Q_i \rtimes H_i$ and $M \simeq Q_i \rtimes G_i$.

By Corollary IV.4.8 (and Remark IV.4.9), we see that $H_i \subset G_i$ satisfies the assumptions of Proposition IV.5.5 so that $N_i \subset M$ has the asymptotic orthogonality property relative to A_i. Moreover the Example IV.5.2 tells us that $N_i \subset M$ is (weakly) mixing through A_i.

By Theorem IV.5.4, one concludes that the amenable algebra P, which contains A_i, is contained in N_i. This ends the proof of Theorem IV.D. □

Chapter V

Questions and perspectives

V.1 Von Neumann algebras associated with Gaussian actions

The main question that attracted my attention during these three years is the following well known conjecture.

Conjecture V.1.1. *For any non-amenable group Γ, the II_1 factor associated with the Bernoulli action $\Gamma \curvearrowright ([0,1],\lambda)^\Gamma$ has a unique Cartan subalgebra, up to unitary conjugacy.*

Following [PV(12)], we say that an action of a group Γ is $\mathcal{C}$-rigid if the associated von Neumann algebra has a unique Cartan subalgebra, up to unitary conjugacy. One can wonder if any non-amenable group has a $\mathcal{C}$-rigid pmp action. Several facts [Po06a, Po06b, AW13] seem to indicate that for a given group, the best candidate for a $\mathcal{C}$-rigid action is the Bernoulli action.

Conjecture V.1.1 is very hard to answer in full generality. I mention here several possible approaches to solve weaker forms of the conjecture and related questions.

First note that Ioana's result [Io11] shows uniqueness of group-measure space Cartan subalgebras whenever Γ has property (T). Would it be possible to find a simpler proof of this result, not relying on the co-product map (see the proof of Theorem II.4.5)?

In view of Popa and Vaes uniqueness of Cartan papers [PV(12), PV(13)], would it be possible to solve Conjecture V.1.1 for non-amenable, weakly amenable groups? More precisely, would it be possible to combine (a variant of) [PV(12), Theorem 5.1] and the Bernoulli deformation to prove uniqueness of Cartan subalgebras in crossed-product factors associated with Bernoulli actions of weakly amenable groups?

Alternatively one could try to use the relative bi-exactness result for Bernoulli actions [BO08, Proposition 15.3.6] instead of the Bernoulli malleable deformation. In the same vein, one can ask the following question.

Question V.1.2. *Consider a non-amenable group Γ and denote by M the crossed-product II_1 factor associated with the Bernoulli action $\Gamma \curvearrowright ([0,1],\lambda)^\Gamma$. Assume that $M \simeq L^\infty(Y,\nu) \rtimes \Lambda$ for a free ergodic pmp action $\Lambda \curvearrowright (Y,\nu)$ of a weakly amenable group Λ.*

Are the Cartan subalgebras $L^\infty([0,1]^\Gamma)$ and $L^\infty(Y)$ necessarily unitarily conjugate?

Furthermore, together with [Pe(09), Lemma 2.6] a positive answer to Question II.3.13 could help to answer the following almost complementary question.

Question V.1.3. *Consider a non-amenable group Γ and denote by M the crossed-product II_1 factor associated with the Bernoulli action $\Gamma \curvearrowright ([0,1],\lambda)^\Gamma$. Assume that $M \simeq L^\infty(Y,\nu) \rtimes \Lambda$ for a free ergodic pmp action $\Lambda \curvearrowright (Y,\nu)$ of a group Λ which does not have the Haagerup property. Are the Cartan subalgebras $L^\infty([0,1]^\Gamma)$ and $L^\infty(Y)$ necessarily unitarily conjugate?*

Finally, if the two questions above have positive answers can one fill the gap to prove uniqueness of group measure space Cartan subalgebras in crossed products associated with Bernoulli actions?

V.2 Problems on maximal amenable subalgebras in free group factors

The work presented in Chapter IV makes use of the Gromov boundary of hyperbolic groups. This boundary is also used in Ozawa's work on solidity and bi-exactness [Oz04, BO08]. It would be interesting to elucidate the link between the two approaches. For instance could we reprove Theorem IV.A using bi-exactness, or property AO of Akemann and Ostrand?

Next, it would be worth further investigating the position of hyperfinite subfactors in free group factors. Theorem IV.A shows that for $n \geq 2$ no element $x \in F_n \subset LF_n$ is contained in a hyperfinite subfactor of LF_n. As was asked by Jean Renault, what about an element $x \in \mathbb{C}F_n \subset LF_n$ or an element $x \in C_r^*(F_n) \subset LF_n$?

Finally, can one produce an explicit example of a hyperfinite subfactor $R \subset LF_n$ such that $R \cap C_r^*(F_n) = \mathbb{C}1$?

Appendix A

(Weakly) Mixing bimodules over finite von Neumann algebras

One of the main concepts in the theory of group representations (and group actions) is the notion of a mixing representation.

In the context of finite von Neumann algebras one can observe various mixing phenomena. In this chapter, we intend to provide a unified approach to the notion of mixing for von Neumann algebras. Then we shall present applications of this notion which are intensively used in Chapters II, III and IV.

A.1 Definitions and examples

Inspired by the notion of mixing representation of a group, Peterson and Sinclair [PS12] introduced the notion of mixing bimodule[1].

Definition A.1.1. Let M and N be von Neumann algebras. An M-N bimodule ${}_M H_N$ is *(left) mixing* if for any bounded sequence of unitaries $(u_n) \subset \mathcal{U}(M)$ which tends weakly to 0, one has

$$\lim_n \sup_{y \in (N)_1} |\langle u_n \xi y, \eta \rangle| = 0, \, \forall \xi, \eta \in H.$$

A more classical definition, though, is the notion of a mixing inclusion of finite von Neumann algebras.

Definition A.1.2. An inclusion of finite von Neumann algebras $N \subset (M, \tau)$ is *mixing* if any sequence $(u_n) \subset \mathcal{U}(N)$ which tends weakly to 0 satisfies

$$\lim_n \|E_N(a u_n b)\|_2 = 0, \, \forall a, b \in M \ominus N.$$

It turns out that mixing inclusions are just a particular case of mixing bimodules.

Proposition A.1.3. *An inclusion $N \subset (M, \tau)$ is mixing if and only if ${}_N(L^2(M) \ominus L^2(N))_N$ is mixing.*

[1] The mixing property of the coarse bimodule was already explicitly used in [Pe09].

Proof. Assume that $N \subset M$ is mixing. Consider a sequence $(u_n) \subset \mathcal{U}(N)$ which tends weakly to 0. By density it is clearly sufficient to check that for all $a, b \in M \ominus N$ we have

$$\lim_n \sup_{y \in (N)_1} |\langle u_n a y, b\rangle| = 0.$$

But for every n we see that

$$\sup_{y \in (N)_1} |\langle u_n a y, b\rangle| = \sup_{y \in (N)_1} |\langle b^* u_n a, y^*\rangle| \leq \|E_N(b^* x_n a)\|_2 \to 0.$$

Conversely assume that the bimodule is left mixing and take a bounded sequence $(u_n) \subset N$ which tends weakly to 0. Take also $a, b \in M \ominus N$. For all n we have

$$\|E_N(a u_n b)\|_2^2 = \langle a u_n b, E_N(a u_n b)\rangle,$$

which tends to 0 because the sequence $E_N(a u_n b)$ is bounded for the operator norm. □

Proposition A.1.4. *A measure preserving action $\Gamma \curvearrowright (X, \mu)$ is mixing if and only if the inclusion $L\Gamma \subset L^\infty(X) \rtimes \Gamma$ is mixing.*

Proof. Denote by $u_g, g \in \Gamma$ the canonical unitaries in M implementing the action of Γ.

First assume that the inclusion is mixing. Then for any sequence $(g_n) \subset \Gamma$ going to infinity and $a, b \in L^\infty(X, \mu) \ominus \mathbb{C}$, we have

$$\lim_n |\tau(a\sigma_{g_n}(b))| = \lim_n \|E_{L\Gamma}(a u_{g_n} b)\|_2 = 0.$$

Conversely, assume that the action is mixing and take a sequence $(u_n) \subset \mathcal{U}(L\Gamma)$ which tends weakly to 0. By linearity and density, it is sufficient to check that for all $a, b \in L^\infty(X, \mu) \ominus \mathbb{C}$ we have $\lim_n \|E_{L\Gamma}(a u_n b)\|_2 = 0$. Fix $a, b \in L^\infty(X, \mu) \ominus \mathbb{C}$ and $\varepsilon > 0$. Writing $u_n = \sum_{g \in \Gamma} \lambda_{n,g} u_g$, we get

$$\|E_{L\Gamma}(a u_n b)\|_2^2 = \sum_{g \in \Gamma} |\lambda_{n,g} \tau(a\sigma_g(b))|^2, \text{for all } n.$$

Since the action is mixing there exists a finite set $F \subset \Gamma$ such that $|\tau(a\sigma_g(b))| \leq \varepsilon$ for all $g \notin F$. Hence for all n, we have

$$\begin{aligned}\|E_{L\Gamma}(a u_n b)\|_2^2 &\leq \sum_{g \in F} |\lambda_{n,g} \tau(a\sigma_g(b))|^2 + \varepsilon^2 \sum_{g \notin F} |\lambda_{n,g}|^2 \\ &\leq \sum_{g \in F} |\lambda_{n,g} \tau(a\sigma_g(b))|^2 + \varepsilon^2.\end{aligned}$$

But since the (u_n) converges weakly to 0, we have that $\lim_n \sum_{g \in F} |\lambda_{n,g} \tau(a\sigma_g(b))|^2 = 0$. This concludes the proof. □

With the same kind of proof, we also get the following examples.

Example A.1.5. Given groups $\Gamma_0 < \Gamma$, the inclusion $L\Gamma_0 \subset L\Gamma$ is mixing if and only if Γ_0 is almost malnormal inside Γ[2].

Example A.1.6. If $M = M_1 * M_2$ is a free product of finite von Neumann algebras, then $M_1 \subset M$ is mixing.

[2]This means that $s\Gamma_0 s^{-1} \cap \Gamma$ is finite for all $s \in \Gamma \setminus \Gamma_0$.

A.2 Weakly mixing bimodules

Definition A.2.1. A bimodule ${}_M H_N$ is (left) weakly mixing if there exists a sequence of unitaries $(u_n) \subset \mathcal{U}(M)$ such that

$$\lim_n \sup_{y \in (N)_1} |\langle u_n \xi y, \eta \rangle| = 0, \ \forall \xi, \eta \in H.$$

From this definition, one deduces corresponding notions of weakly mixing inclusions of finite von Neumann algebras, and provide examples similar to what we did for mixing bimodules.

In the context of group representations, recall that we have an equivalence between the three statements:

- a representation π is weakly mixing;
- π has no (non-zero) finite dimensional sub-representation;
- $\pi \otimes \pi$ has no (non-zero) invariant vectors.

The following theorem is the von Neumann algebraic analogue of this statement. The proof is similar to the proof of Popa's intertwining lemma [Po06a, Po06d]. This is not a coincidence, as we will see in the next section. Here ${}_N \overline{H}_M$ denotes the contragredient bimodule of the bimodule ${}_M H_N$, and $H \otimes_N \overline{H}$ is their fusion product in the sense of Connes. For more on this, see [Co94, Fa09].

Theorem A.2.2. *Assume that ${}_M H_N$ is a bimodule over finite von Neumann algebras M and N. The following are equivalent:*

(i) ${}_M H_N$ *is weakly mixing;*

(ii) $\{0\}$ *is the only M-N subbimodule of ${}_M H_N$ which has finite N-dimension;*

(iii) ${}_M(H \otimes_N \overline{H})_M$ *has no non-zero central vector.*

Proof. $(i) \Rightarrow (iii)$. Assume that ${}_M H_N$ is weakly mixing. Then this is also the case of the bimodule ${}_M(H \otimes_N \overline{H})_M$, which can not contain a central vector.

$(iii) \Rightarrow (ii)$. Assume that ${}_M(H \otimes_N \overline{H})_M$ has no central vector.

Claim. $\{0\}$ is the only subbimodule of ${}_M H_N$ which is finitely generated as an N-module.

By contradiction, consider a non-zero subbimodule ${}_M K_N \subset {}_M H_N$ which is finitely N-generated: there exists an onto isomorphism of right modules $u : K_N \to (pL^2(N)^{\oplus n})_N$ for some $n \geq 1$ and some projection $p \in M_n(\mathbb{C}) \otimes N$, $p \neq 0$.

Then the left M-action on K induces a $*$-homomorphism $\varphi : M \to p(M_n(\mathbb{C}) \otimes N)p$ such that $u(x\xi) = \varphi(x)u(\xi)$ for all $x \in M$, $\xi \in K$.

For all $j = 1, \cdots, n$, denote by $\xi_j = u^{-1}(pe_j)$, where $e_j \in L^2(N)^{\oplus n}$ is the j^{th} coordinate vector $e_j = (0, \cdots, 0, 1, 0, \cdots, 0)$. One checks that the vectors ξ_i are right N-bounded and satisfy the relations

$$x\xi_i = \sum_{j=1}^n \xi_j \cdot (\varphi(x))_{j,i}, \text{ for all } x \in M, i = 1, \cdots, n.$$

Then the vector $\xi := \sum_{k=1}^n \xi_k \otimes \overline{\xi_k}$ is an M-central vector in $K \otimes_N \overline{K} \subset H \otimes_N \overline{N}$. Indeed, for all $x \in M$ we have

$$\begin{aligned} x \cdot \xi = \sum_{i=1}^n (x \cdot \xi_i) \otimes \overline{\xi_i} &= \sum_{i,j=1}^n (\xi_j \cdot (\varphi(x))_{j,i}) \otimes \overline{\xi_i} \\ &= \sum_{i,j=1}^n \xi_j \otimes \overline{\xi_i \cdot (\varphi(x))_{j,i})^*} \\ &= \sum_{i,j=1}^n \xi_j \otimes \overline{\xi_i \cdot (\varphi(x^*))_{i,j})} = \sum_{j=1}^n \xi_j \otimes \overline{x^* \cdot \xi_j} = \xi \cdot x. \end{aligned}$$

But since the vectors ξ_i, $i = 1, \cdots, n$ are pairwise orthogonal, the vector ξ is non-zero. This contradicts our assumption (iii).

Now that the claim is proved, assume that K is a non-zero subbimodule of ${}_M H_N$ with finite N-dimension. Denote by $1 - p \in \mathcal{Z}(N)$ the maximal projection in $\mathcal{Z}(N)$ such that $Kp = \{0\}$. By assumption, $1 - p \neq 1$ and so $p \neq 0$. Now consider the bimodule ${}_M(Kp)_{pN}$. It has finite right dimension, and so there exists a non-zero projection $q \in \mathcal{Z}(pN)$ such that $(Hq)_{qN}$ is finitely generated (see [Va07, Lemma A.1]). Therefore ${}_M(Kq)_{qN}$ is a non-zero subbimodule of the weakly mixing bimodule ${}_M(Hq)_{qN}$ which is finitely N-generated. This contradicts the claim.

$(ii) \Rightarrow (i)$. Assume that ${}_M H_N$ is not weakly mixing. Denote by $H^0 \subset H$ the (dense) subspace of right bounded vectors, and for $\xi \in H^0$ denote by $L_\xi : L^2(M) \to K$ the operator defined by $L_\xi x = \xi x$ for all $x \in N$. Recall that for all $\xi, \eta \in H^0$, and any $x \in M$ we have $L_\xi^* x L_\eta \in N \subset B(L^2(N))$.

Since ${}_M H_N$ is not weakly mixing there exists $\varepsilon > 0$ and a finite set $F \subset H^0$ such that for all $x \in M$ we have

$$\sum_{\xi,\eta \in F} |\sup_{y \in (N)_1} \langle x\xi y, \eta\rangle|^2 \geq \varepsilon.$$

Equivalently, for all $x \in M$ we have

$$\sum_{\xi,\eta \in F} \|L_\xi^* x L_\eta\|_1^2 \geq \varepsilon.$$

Now define an element $c \in B(H) \cap (N^{\mathrm{op}})'$ by the formula $c = \sum_{\xi \in F} L_\xi L_\xi^*$.

Denote by Tr the canonical semi-finite faithful normal trace on $B(H) \cap (N^{\mathrm{op}})'$ which satisfies $\mathrm{Tr}(L_\xi L_\eta^*) := \tau(L_\eta^* L_\xi)$ for all $\xi, \eta \in H^0$. We see that $\mathrm{Tr}(c)$ is finite.

Consider the ultraweakly closed convex hull $\mathcal{C}$ of the set $\{ucu^* \,|\, u \in \mathcal{U}(M)\}$. Then $\mathcal{C}$ is a closed convex set in the Hilbert space $L^2(B(H) \cap (N^{\mathrm{op}})', \mathrm{Tr})$. So it admits a unique element $d \in \mathcal{C}$ of minimal $\|\cdot\|_{2,\mathrm{Tr}}$-norm. By uniqueness, and since $\mathcal{C}$ is invariant under conjugacy by $\mathcal{U}(M)$ we get that $d \in M' \cap (N^{\mathrm{op}})'$. Let us show that $d \neq 0$. For all $u \in \mathcal{U}(M)$ we have

$$\begin{aligned} \sum_{\eta \in F} \tau(L_\eta^* ucu^* L_\eta) &= \sum_{\xi,\eta \in F} \tau(L_\eta^* u L_\xi L_\xi^* u^* L_\eta) \\ &= \sum_{\xi,\eta \in F} \|L_\xi^* u^* L_\eta\|_2^2 \\ &\geq \sum_{\xi,\eta \in F} \|L_\xi^* u^* L_\eta\|_1^2 \geq \varepsilon. \end{aligned}$$

By continuity, we get that $\sum_{\eta\in F}\tau(L_\eta^* d L_\eta) \geq \varepsilon$, so that $d \neq 0$.

Take a non-zero spectral projection $p \in M' \cap (N^{\mathrm{op}})'$ of d^*d. We have $\mathrm{Tr}(p) \leq \|d\|_{2,\mathrm{Tr}}^2 < \infty$. This precisely means that pH is a subbimodule of ${}_M H_N$ with finite right N-dimension. □

Corollary A.2.3. *Assume that $N \subset (M,\tau)$ is a mixing inclusion of finite von Neumann algebras. Then for any diffuse subalgebra $Q \subset N$, we have $\mathcal{QN}_M(Q) \subset N$.*

Proof. By assumption, we know that ${}_N(L^2(M) \ominus L^2(N))_N$ is mixing. Since $Q \subset N$ is diffuse, the bimodule ${}_Q(L^2(M) \ominus L^2(N))_Q$ is weakly mixing. Assuming that $v \in \mathcal{QN}_M(Q)$, we get that $x := v - E_N(v) \in \mathcal{QN}_M(Q)$ as well. Therefore $\overline{\mathrm{span}}(QxQ) \subset L^2(M) \ominus L^2(N)$ is a subbimodule with finite right B-dimension so it has to be $\{0\}$ by Theorem A.2.2. Hence $v = E_N(v)$. □

A.3 Popa's intertwining-by-bimodules lemma

Since its first developments in the early 2000's, Popa's deformation/rigidity theory has lead to numerous breakthroughs. Together with the concepts of deformation and rigidity came a very powerful technical tool: the so-called "intertwining-by-bimodules lemma" discovered by Popa [Po06a]. This lemma is certainly the key to the success of Deformation/rigidity theory.

Definition A.3.1. Let $A, B \subset (M,\tau)$ be finite von Neumann algebras (with possibly non-unital inclusions). We say that A *embeds into* B *inside* M (and we write $A \prec_M B$) if the bimodule ${}_A(1_A L^2(M) 1_B)_B$ is not weakly mixing.

Remark A.3.2. Proceeding as in the proof of Proposition A.1.3, one can check that $A \prec_M B$ if and only if there is no sequence of unitaries $(u_n) \subset \mathcal{U}(A)$ such that

$$\lim_n \|E_B(1_B x u_n y 1_B)\|_2 = 0, \, \forall x, y \in M.$$

The terminology is justified by the following theorem. The equivalence between (i) and (ii) below is due to Theorem A.2.2. We also provided the material to prove the equivalence with (iii) in the proof of Theorem A.2.2. We refer to the original articles [Po06a, Po06d], or to [BO08, Appendix F] for more details.

Theorem A.3.3 (Intertwining-by-bimodules, [Po06a, Po06d])**.** *Let $A, B \subset (M,\tau)$ be finite von Neumann algebras (with possibly non-unital inclusions). Then the following are equivalent.*

- *(i)* $A \prec_M B$;
- *(ii) There exists a subbimodule of ${}_A L^2(M)_B$ with finite right B-dimension;*
- *(iii) There exist projections $p \in A$, $q \in B$, a normal $*$-homomorphism $\psi : pAp \to qBq$, and a non-zero partial isometry $v \in pMq$ such that $xv = v\psi(x)$, for all $x \in pAp$;*

Example A.3.4. Assume that M is of the form $M = B \rtimes \Gamma$ for some trace preserving action of Γ on a finite von Neumann algebra and B. Denote by $u_g, g \in \Gamma$ the canonical unitaries in M implementing the action of Γ.

A subalgebra $A \subset M$ satisfies $A \not\prec B$ if and only if there exists a sequence of unitaries $v_n \in \mathcal{U}(A)$ such that

$$\|E_B(v_n u_g^*)\|_2 \to 0, \, \forall g \in \Gamma.$$

If B is abelian in the above example and if the action is free and ergodic, then B is a Cartan subalgebra of the factor M. The following result states that for Cartan subalgebras intertwining-by-bimodules amounts to unitary conjugacy.

Theorem A.3.5 ([Po06c], Theorem A.1)**.** *Assume that A and B are Cartan subalgebras in a II_1-factor M. Then A embeds into B inside M if and only if there exits $u \in \mathcal{U}(M)$ such that $uAu^* = B$.*

Proposition A.3.5 is a real motivation to further study properties of the relation $\prec_M$. We refer to [Va08, Section 3] for elementary stability properties (relative commutant, amplification/reduction...) and for a discussion on the transitivity of this relation.

The following criterion, due to Ioana, is an improvement of the characterization given in Example A.3.4. It will play a crucial role in the proof of Theorem II.3.1.

Proposition A.3.6 ([Io11], Theorem 1.3.2)**.** *Let $\Gamma \curvearrowright B$ be a trace preserving action on a finite von Neumann algebra (B, τ). Put $M = B \rtimes \Gamma$, and let $P \subset M$ be a von Neumann subalgebra. Then $P \not\prec B$ if and only if there exists a sequence of unitaries $v_n \in \mathcal{U}(P)$ such that*

$$\lim_n \left(\sup_{g \in \Gamma} \|E_B(v_n u_g^*)\|_2 \right) = 0.$$

A.4 Relatively mixing bimodules

We define a relative version of mixing bimodules, more adapted to our purposes. Let us start with a convenient definition.

Definition A.4.1. Consider finite von Neumann algebras $A \subset (M, \tau)$. We say that a bounded sequence $(x_n) \subset M$ *(weakly) tends to* 0 *relative to* A if $\lim_n \|E_A(a x_n b)\|_2 = 0$ for all $a, b \in M$.

If $A = \mathbb{C}$, this amounts to saying that (x_n) tends weakly to 0.

Definition A.4.2. Let $A \subset (M, \tau)$ and N be finite von Neumann algebras. A bimodule ${}_M H_N$ is *mixing relative to* A if for any sequence of unitaries $(u_n) \subset \mathcal{U}(M)$ which tends to 0 relative to A, one has

$$\lim_n \sup_{y \in (N)_1} |\langle u_n \xi y, \eta \rangle| = 0, \forall \xi, \eta \in H.$$

Definition A.4.3. An inclusion $N \subset (M, \tau)$ is mixing relative to a subalgebra A of N if the bimodule ${}_N(L^2(M) \ominus L^2(N))_N$ is mixing relative to A.

Example A.4.4. As in Section A.1 we get very concrete examples.

1. Consider a measure preserving action $\Gamma \curvearrowright^\sigma (X, \mu)$ and any trace preserving action $\Gamma \curvearrowright (A, \tau)$. Then $(1 \otimes A) \rtimes \Gamma \subset (L^\infty(X, \mu) \overline{\otimes} A) \rtimes \Gamma$ is mixing relative to $1 \otimes A$ if and only if σ is mixing.

2. Consider a trace-preserving action $\Gamma \curvearrowright (A, \tau)$ and take a subgroup $\Gamma_0 < \Gamma$. The inclusion $A \rtimes \Gamma_0 \subset A \rtimes \Gamma$ is mixing relative to A if and only if Γ_0 is almost malnormal inside Γ.

3. If $M = M_1 *_A M_2$ is an amalgamated free product of finite von Neumann algebras, then $M_1 \subset M$ is mixing relative to A.

Proposition A.4.5. *Consider finite von Neumann algebras $A \subset N \subset (M,\tau)$. Assume that $N \subset M$ is mixing relative to A. Then for any $Q \subset N$ such that $Q \not\prec A$, we have that $\mathcal{QN}_M(Q) \subset N$.*

Proof. Since $Q \not\prec_M A$, we get that ${}_Q(L^2(M) \ominus L^2(N))_Q$ is weakly mixing. So the conclusion is a consequence of Theorem A.2.2. $\square$

Proposition A.4.6. *Assume that $A \subset N \subset M$ are finite von Neumann algebras such that $N \subset M$ is mixing relative to A. Take a projection $p \in M$ and a subalgebra $Q \subset pMp$ such that $Q \not\prec_M A$. Put $P = \mathcal{QN}_{pMp}(Q)''$.*

1. *If $Q \prec_M N$ then there exists a non-zero partial isometry $v \in pM$ such that $vv^* \in P$ and $v^*Pv \subset N$.*
2. *If moreover N is a factor then one can find such a v with $vv^* \in \mathcal{Z}(P)$.*
3. *If N is a factor and if $rQ \prec_M N$ for all $r \in Q' \cap pMp$ then there exists a unitary $u \in \mathcal{U}(M)$ such that $uPu^* \subset N$.*

Proof. (1) By assumption, there exist projections $p_0 \in Q$, $q \in N$, a non-zero partial isometry $v \in p_0Mq$ and a *-homomorphism $\varphi : p_0Qp_0 \to qNq$ such that for all $x \in p_0Qp_0$, one has $xv = v\varphi(x)$.

By [Va08, Remark 3.8], one can assume that $\varphi(p_0Qp_0) \not\prec_M A$. Hence Proposition A.4.5 implies that $\mathcal{QN}_{qMq}(\varphi(p_0Qp_0))'' \subset N$. But we see that $v^*Pv \subset \mathcal{QN}_{qMq}(\varphi(p_0Qp_0))''$. Moreover $vv^* \in p_0(Q' \cap M) \subset P$.

(2) Let us modify v obtained above in such a way that $vv^* \in \mathcal{Z}(P)$.

Take partial isometries $v_1, \cdots, v_k \in P$ such that $v_i^*v_i \leq vv^*$, $i = 1, \cdots, k$ and $\sum_{i=1}^k v_iv_i^*$ is a central projection in P. Since N is a factor, there exist partial isometries $w_1, \cdots, w_k \in N$ such that $w_iw_i^* = v^*v_i^*v_iv$ and $w_iw_j^* = 0$, for all $1 \leq i \neq j \leq k$. Define a non-zero partial isometry by $w = \sum_i v_ivw_i \in pM$. We get

- $ww^* = \sum_i v_ivw_iw_i^*v^*v_i^* = \sum_i v_iv_i^* \in \mathcal{Z}(P)$;
- $w^*Pw \subset \sum_i w_i^*v^*Pvw_i \subset N$.

(3) Consider a maximal projection $r_0 \in \mathcal{Z}(P)$ for which there exists a unitary $u \in \mathcal{U}(M)$ such that $u(r_0P)u^* \subset N$. One has to show that $r_0 = p$. Otherwise we can cut by $r = p - r_0$, and we obtain an algebra $rQ \subset rMr$ such that $rQ \prec_M N$ and $rQ \not\prec_M A$. Remark that $rP \subset \mathcal{QN}_{rMr}(rQ)''$. Applying (2), we get that there exists a non-zero partial isometry $v \in rM$, such that $vv^* \in \mathcal{Z}(rP)$ and $v^*(rP)v \subset N$.

Since N is a factor, modifying v if necessary, one can assume that $v^*v \perp ur_0u^*$. Now the following "cutting and pasting" argument contradicts the maximality of r_0. The partial isometry $w_0 = ur_0 + v^*$ satisfies $w_0^*w_0 = r_0 + vv^* \in \mathcal{Z}(P)$ and $w_0(r_0 + vv^*)Pw_0^* \subset N$. Extending w_0 into a unitary, we obtain a $w \in \mathcal{U}(M)$ satisfying $w(r_0 + vv^*)Pw^* \subset N$. $\square$

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